

F 1503 / 1

**BULETINUL
INSTITUTULUI
POLITEHNIC
DIN IAȘI**

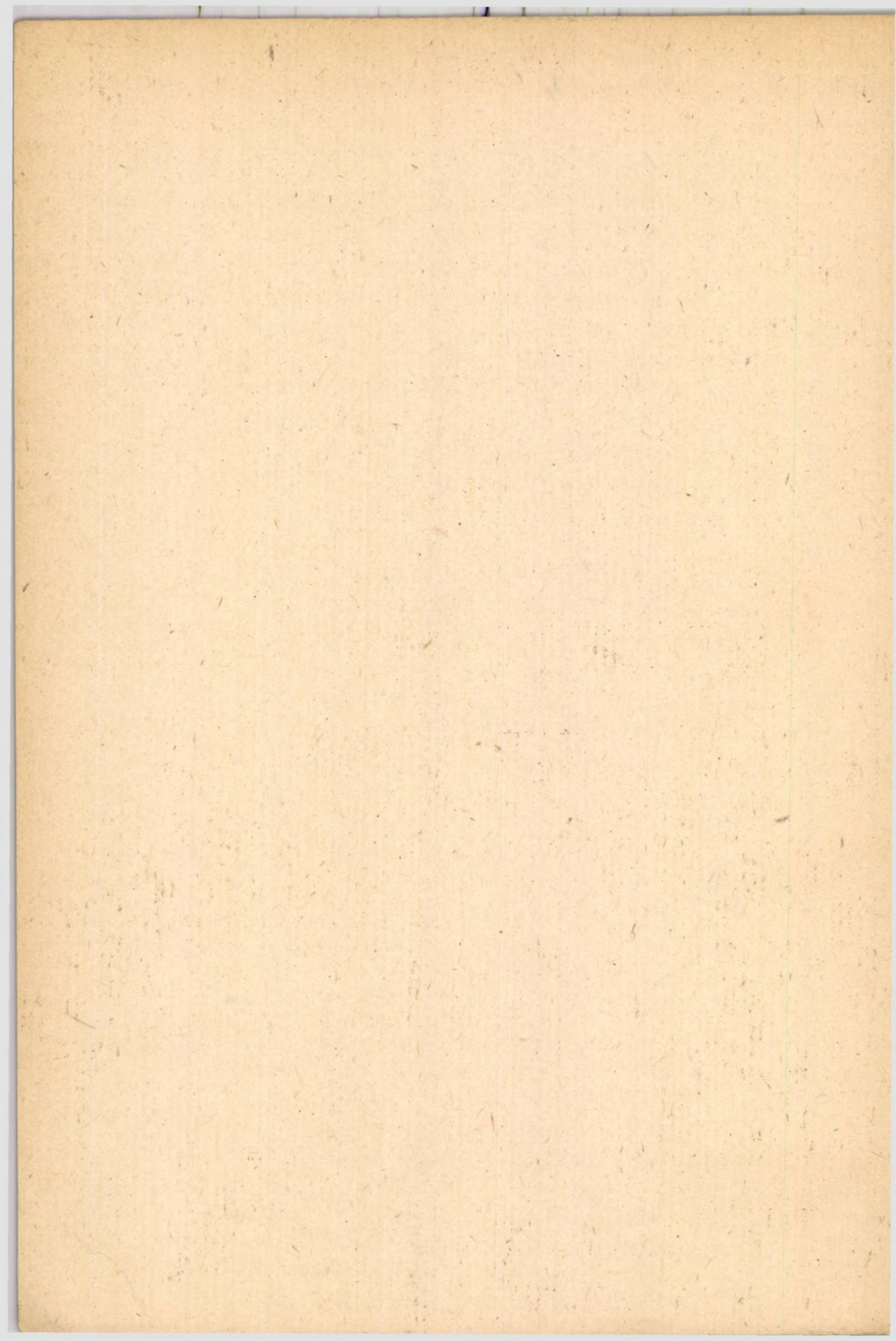
Tomul XXXVIII (XLII)

Fasc. 1—4

Secția I

**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1992





BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI

Tomul XXXVIII (XLII)

Fasc. 1—4

Secția I

**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1992

EDITORIAL SUAFF

Prof. dr. ing. **VITALIE BELOUSOV** (President)

Prof. dr. doc. **D. I. MANGERON** (Honorary President)

Conf. dr. ing. **MIRCEA MANOLOVICI** (Secretary)

Conf. dr. ing. **MATEI MACOVEANU**

Prof. dr. doc. ing. **ALFRED BRAIER** (Editor)

Prof. dr. ing. **HUGO ROSMAN** (Editor)

Prof. dr. ing. **MIHAIL VOICU** (Head of the Exchange Service)

Prof. dr. mat. **AUREL BEJANCU**

Prof. dr. ing. **VICTOR BULACOVSCI**

Prof. dr. ing. **MIHAI LUCANU**

Prof. dr. ing. **SPIRIDON CREȚU**

Prof. dr. ing. **ADRIAN RADU**

Conf. dr. ing. **AURELIA GRIGORIU**

Secția I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

S U M A R

	Pag.
ADRIAN CORDUNEANU, O metodă directă pentru forma Jordan a matricelor de ordinul patru (engl., rez. rom.)	1
GH. COSTOVICI, Unele grupuri ordonate pe unele produse carteziene (II) (engl., rez. rom.)	7
P. TAMIA-DIMOPOULOU (Grecia), Asupra geometriei pe fibratul vectorial vertical (engl., rez. rom.)	11
CĂTĂLINA NIȚESCU, Conexiunea Kawaguchi metrică a unei conexiuni liniare (engl., rez. rom.)	19
C. CĂLIN, CR-subvarietăți ale varietăților aproape de contact metrice normale (engl., rez. rom.)	23
TEMISTOCLE BÎRSAN, Cîteva considerații asupra spațiilor T -metrice compacte (franc., rez. rom.)	27
GH BANTAȘ, Despre admisibilitatea unei perechi de spații de funcții riglate în raport cu un operator integral n. linear de tip Volterra (franc., rez. rom.)	31
C. SIMIRAD, Asupra sistemelor de ecuații diferențiale cu parametru mic (franc., rez. rom.)	35
I. ONCIULESCU, Soluții periodice și oscilatorii ale unui model de populație dependent de vîrstă (engl., rez. rom.)	41
GEORGETA TEODORU, Selecții continue pentru aplicații multivoce cu membrul drept neconvex și problema lui Goursat pentru ecuația hiperbolică multivocă $\partial^2 z / \partial x \partial y \in F(x, y, z)$ (engl., rez. rom.)	49
S. D. BAJPAI (Bahrain), Transmiterea căldurii într-o regiune infinită și polinoamele Hermite (engl., rez. rom.)	57
VIOREL GHIUȘ, Transformări ale unei circulații standard admisibile dintr-un graf latice cu o dimensiune (franc., rez. rom.)	59
I. IBĂNESCU, Asupra deformării plăcilor parabolice (engl., rez. rom.)	63
D. VIERU și SILVIA-GABRIELA CIUMAȘU, Teoreme de reciprocitate pentru problema dinamică a termoeleasticității cuplate generalizate cu microstructură și unele aplicații (engl., rez. rom.)	69

D. MANOLE, Teoreme variaționale în teoria viscoelasticității liniare micropolare (engl., rez. rom.)	75
CORNELIU D. CIUBOTARIU și ILIE BURDUJAN, Metoda scalei multiple de timp și construcția unui „superceas” (engl., rez. rom.)	85
GH. ZET, MARCEL AGOP, C. PASNICU, C. BRUMĂ, C. DĂRIESCU și MARINA DĂRIESCU, Un model de gravitație pe sfera $R \times S^3$ (engl., rez. rom.)	93
V. MELNIC, MARCEL AGOP, I. SANDU, MARIANA CAIA, MARINA DĂRIESCU și P. DURBACA, Cîteva semnificații fizice pe varietatea grupului unimodular G_3 (franc., rez. rom.)	95
I. SANDU, MARCEL AGOP, GH. ZET, V. MELNIC, NICULINA DURBACA, MARIANA CAIA, C. DĂRIESCU și VIOLETA GHIUTCAN, Noi aspecte în mecanica invariantivă (engl., rez. rom.)	99
MARCEL AGOP, C. DĂRIESCU și MARINA DĂRIESCU, Ecuațiile Lagrange, Hamilton și Hamilton-Jacobi pe varietatea $E^3 \times T$ (franc., rez. rom.)	103
EMIL LUCA și R. BĂDESCU, Asupra propagării ultrasunetelor în ferrofluide (engl., rez. rom.)	107
EUGEN NEAGU, RODICA NEAGU și CLAUDIA BOTEZ, Corelația între poziția maximelor TSD și condițiile de formare și stocare (engl., rez. rom.)	111
MARIANA ISTRATE și SERGIU ISTRATE, Fenomenul de antrenare și efectul Hall în semiconductori cu benzi neuniforme (engl., rez. rom.)	117
IRINA ANTONESCU, M. ANTONESCU, C. TĂNĂSOIU și POMPILIU NICOLAU, Efectele variațiilor temperaturii asupra comportării piezoelectrice a zircotitanaților de plumb modificați, al căror raport Zr/Ti variază în limitele tranziției morfotropice de fază (franc., rez. rom.)	125

Section I

MATHEMATICS. THEORETICAL MECANICS. PHYSICS

C O N T E N T S

	<u>Pp.</u>
ADRIAN CORDUNEANU, A Direct Method for the Jordan Form of the Fourth Order Matrices (English, Romanian summary)	1
GH. COSTOVICI, Some Ordered Groups on Some Cartesian Products (II) (English, Romanian summary)	7
P. TAMIA-DIMOPOULOU (Greece), On Geometry of Vertical Vector Bundle (English, Romanian summary)	11
CĂTĂLINA NIȚESCU, Metrical Kawaguchi Connection of a Linear Connection (English, Romanian summary)	19
C. CĂLIN, Semi-Invariant Submanifolds of a Normal Almost Contact Metric Manifold (English, Romanian summary)	23
TEMISTOCLE BÎRSAN, Quelques considérations sur les espaces T -métriques compacts (French, Romanian summary)	27
GH. BANTAȘ, Sur l'admissibilité d'une paire d'espaces de fonctions réglées par rapport à un opérateur intégral non linéaire de type Volterra (French, Romanian summary)	31
C. SIMIRAD, Sur les systèmes d'équations différentielles à petit paramètre (French, Romanian summary)	35
I. ONCIULESCU, Periodical and Oscillatory Solutions of an Age Dependent Population Model (English, Romanian summary)	41
GEORGETA TEODORU, Continuous Selections of Multi-Valued Maps with Non-Convex Right-Hand Side and the Goursat Problem for the Multi-Valued Hyperbolic Equation $\partial^2 z / \partial x \partial y \in F(x, y, z)$ (English, Romanian summary)	49
S. D. BAJPAI (Bahrain), Heat Conduction in an Infinite Region and Hermite Polynomials (English, Romanian summary)	57
VIOREL GHIUȘ, Transformations de la circulation standard admissible dans un graphe latticiel avec une dimension non orientée (French, Romanian summary)	59
I. IBĂNESCU, On the Deformation of Parabolical Shells (English, Romanian summary)	63

D. VIERU and SILVIA-GABRIELA CIUMAȘU, Reciprocity Theorems for the Dynamic Problem of Coupled Generalized Thermoelasticity with Microstructure and Some Applications (English, Romanian summary)	69
D. MANOLE, Variational Theorems in Linear Theory of Micropolar Viscoelasticity (English, Romanian summary)	75
CORNELIU D. CIUBOTARIU and ILIE BURDUJAN, Multiple Time Scale Method and the Construction of a „Superclock“ (English, Romanian summary)	85
GH. ZET, MARCEL AGOP, C. PASNICU, C. BRUMĂ, C. DĂRIESCU and MARINA DĂRIESCU, A Model for Gravitation on the $R \times S^3$ Topology (English, Romanian summary)	93
V. MELNIC, MARCEL AGOP, I. SANDU, MARIANA CAIA, MARINA DĂRIESCU et P. DURBACA, Quelques significations physiques sur la variété du groupe unimodulaire G_3 (French, Romanian summary)	95
I. SANDU, MARCEL AGOP, GH. ZET, V. MELNIC, NICULINA DURBACA, MARIANA CAIA, C. DĂRIESCU and VIOLETA GHIURCAN, New Aspects in the Invariantive Mechanics (English, Romanian summary)	99
MARCEL AGOP, C. DĂRIESCU et MARINA DĂRIESCU, Les équations Lagrange, Hamilton et Hamilton-Jacobi sur la variété $E^3 \times T$ (French, Romanian summary)	103
EMIL LUCA and R. BĂDESCU, On the Ultrasonic Propagation in Ferrofuids (English, Romanian summary)	107
EUGEN NEAGU, RODICA NEAGU and CLAUDIA BOTEZ, The Correlation Between the Position of the TSD Peaks and the Formation and Storage Conditions (English, Romanian summary)	111
MARIANA ISTRATE and SERGIU ISTRATE, The Drag Phenomenon and the Hall Effect in Semiconductors with Nonuniform Bands (English, Romanian summary)	117
IRINA ANTONESCU, M. ANTONESCU, C. TĂNĂSOIU et POMPILIU NICOLAU, Les effets des variations de la température sur le comportement piézoélectrique des zirco-titanates de plomb céramiques avec des modifications, dont le rapport Zr/Ti varie dans le domaine de la transition morphotropique de phase (French, Romanian summary)	125

A DIRECT METHOD FOR THE JORDAN FORM OF THE FOURTH ORDER MATRICES

BY

ADRIAN CORDUNEANU

The aim of this Note is to obtain the Jordan form (J. f.) for the matrices of the fourth order, whose entries are real or complex numbers, by an elementary way, without making use of the corresponding general theorem, which is difficult enough to be proved. In what follows, we only need some well-known propositions [1] from the matrix theory, which are independent of the Jordan theorem.

Lemma 1 (The Cayley-Hamilton theorem). *If $P(\lambda) = \det(A - \lambda I_n)$ is the characteristic polynomial of the n -th order matrix A , then $P(A) = 0$.*

Lemma 2 (The Sylvester inequalities). *If A, B are n -th order matrices and $r(A)$ = the rank of A , then $r(A) + r(B) - n \leq r(AB) \leq r(A), r(B)$.*

Lemma 3. *If $P(\lambda)$ is the characteristic polynomial of the n -th order matrix A and if $p \geq 1$ is the multiplicity of the root $\lambda = \lambda_1$ in the equation $P(\lambda) = 0$, then $r(A - \lambda_1 I_n) \geq n - p$.*

Throughout this paper, A is an arbitrary 4×4 matrix and I is the identity matrix of order 4. If $P(\lambda)$ has distinct roots $\lambda_i, i = \overline{1, 4}$ and if $v_i, i = \overline{1, 4}$ are the corresponding eigenvectors of A , we have $H^{-1}AH = J$, where $H = \text{col}[v_1, v_2, v_3, v_4]$ and $J = \text{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, consequently A may be diagonalized. The same conclusion still holds in the case $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$ and $r(A - \lambda_1 I) = 2$, because for the double eigenvalue $\lambda = \lambda_1$, we have two linearly independent (l.i.) vectors v_1 and v_2 . In the next case, we encounter a different situation.

Proposition 1. *Assume that $P(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)(\lambda - \lambda_3)$ and $r(A - \lambda_1 I) = 3$. Then, the J. f. of matrix A is*

$$(1) \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \text{ where } J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}, J_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{bmatrix}.$$

Proof. Of course, it was supposed that $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Denote $B = (A - \lambda_2 I)(A - \lambda_3 I)$; because $r(A - \lambda_2 I) = r(A - \lambda_3 I) = 3$, we have $r(B) \geq 2$, consequently B has two l. i. columns.

Taking into account that $\lambda = \lambda_1$ has a unique eigenvector (neglecting the factor of proportionality), we conclude that there exists a column of B , denoted v_2 , such that $(A - \lambda_1 I)v_2 = v_1 \neq \theta = \text{the null vector of type } 4 \times 1$. We

have $(A - \lambda_1 I)v_1 = (A - \lambda_1 I)^2 v_2 = \theta$, because $(A - \lambda_1 I)^2 B = 0$ (Lemma 1). Resuming, we can state that there exist four l. i. vectors v_i , $i = \overline{1, 4}$ such that

$$(2) \quad Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2 + v_1, \quad Av_3 = \lambda_2 v_3, \quad Av_4 = \lambda_3 v_4,$$

which implies $AH = HJ$, where $H = \text{col}[v_1, v_2, v_3, v_4]$ and J is given by (1). This ends the proof.

Remark 1. We point out that the J. f. of A may be easily obtained, firstly solving the system $Av_1 = \lambda_1 v_1$; with v_1 fixed, we shall search v_2 from the system $(A - \lambda_1 I)v_2 = v_1$, whose compatibility is ensured by the preceding Proposition 1, etc. Consequently, in a concrete case, it is not necessary to consider the above cited matricial relations and to make use of Lemmas 1, 2, 3. All the problem consists only in solving some linear systems of equations.

We now assume that $P(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)^2$, with $\lambda_1 \neq \lambda_2$. If we suppose $r(A - \lambda_1 I) = r(A - \lambda_2 I) = 2$, we find four l. i. vectors v_i , $i = \overline{1, 4}$ such that $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_1 v_2$, $Av_3 = \lambda_2 v_3$, $Av_4 = \lambda_2 v_4$ and the matrix A may be diagonalized. A more difficult situation appears in the following

Proposition 2. Assume that $P(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)^2$ and that $r(A - \lambda_1 I) = 2$, $r(A - \lambda_2 I) = 3$. Then, the J. f. of matrix A is

$$(3) \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \text{where } J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}.$$

Proof. We find two l. i. vectors v_1 and v_2 such that $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_1 v_2$ and only a single vector $v_3 \neq \theta$ (if neglect a factor of proportionality) such that $Av_3 = \lambda_2 v_3$. Suppose that $(A - \lambda_2 I)(A - \lambda_1 I)^2 \neq 0$; then, there exists a column $w \neq \theta$ in $(A - \lambda_1 I)^2$, such that $(A - \lambda_2 I)w = v \neq \theta$. We have $(A - \lambda_2 I)v = (A - \lambda_2 I)^2 w = \theta$, if remember that $(A - \lambda_2 I)^2(A - \lambda_1 I)^2 = 0$ (Lemma 1). Thus, we may assume $v_3 = v$ and if we put $v_4 = w$, it follows

$$(4) \quad Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2, \quad Av_3 = \lambda_2 v_3, \quad Av_4 = \lambda_2 v_4 + v_3,$$

which implies $AH = HJ$, where $H = \text{col}[v_1, v_2, v_3, v_4]$, J is given by (3) and the conclusion holds. For the second part of the proof, we firstly remark that $(A - \lambda_2 I)(A - \lambda_1 I) \neq 0$. Indeed, if suppose $(A - \lambda_2 I)(A - \lambda_1 I) = 0$, from the fact that $A - \lambda_1 I$ has two l. i. columns, we would obtain that for $\lambda = \lambda_2$ it correspond two l. i. eigenvectors (these columns) which is a contradiction, because $r(A - \lambda_2 I) = 3$. We now suppose that $(A - \lambda_2 I)(A - \lambda_1 I)^2 = 0$. Let w be a column of $A - \lambda_1 I$, for which $v = (A - \lambda_1 I)w \neq \theta$. Such vector w exists, because $(A - \lambda_1 I)^2 \neq 0$; the assumption that $(A - \lambda_1 I)^2 = 0$ leads us to the conclusion that $(\lambda - \lambda_1)^2$ is an anulant polynomial for A , which is a contradiction. Obviously we have $(A - \lambda_2 I)v = \theta$ and hence if we put $v_3 = v$, $v_4 = w$ we again obtain (4), which concludes the proof.

In the case $P(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)^2$, we also have the following

Proposition 3. Assume that $P(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)^2$ and that $r(A - \lambda_1 I) = 3$, $r(A - \lambda_2 I) = 3$. Then, the J. f. of matrix A is

$$(5) \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \text{where } J_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}.$$

Proof. Denoting $B = (A - \lambda_2 I)^2$, we have $r(B) \geq 2$, hence B has at least two l. i. columns. Since the eigenvalue $\lambda = \lambda_1$ has only one eigenvector,

it follows that there exists a column v_2 of B , such that $(A - \lambda_1 I)v_2 = v_1 \neq \theta$. We have $(A - \lambda_1 I)v_1 = \theta$, using the same argument as and in the proof of Proposition 1. Thus, we have $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_1 v_2 + v_1$ and, in the same way, we find two nonzero vectors v_3 and v_4 such that $Av_3 = \lambda_2 v_3$, $Av_4 = \lambda_2 v_4 + v_3$. Because v_i , $i = \overline{1, 4}$ are l. i. (exercice) and $AH = HJ$, where J is given by (5) and $H = \text{col}[v_1, v_2, v_3, v_4]$, the conclusion easily holds. We point out again that to obtain these four vectors, it only suffices to solve some linear algebraic systems (four equations with four unknowns), without making use of more sophisticated notions from the theory of linear operators.

Now, let us consider the case $P(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$, $\lambda_1 \neq \lambda_2$. If $r(A - \lambda_1 I) = 1$, we find three l. i. eigenvectors v_i , $i = \overline{1, 3}$ corresponding to the eigenvalue $\lambda = \lambda_1$, the J. f. of the matrix A being $J = \text{diag}[\lambda_1, \lambda_1, \lambda_1, \lambda_2]$; this case is not interesting from our point of view.

Proposition 4. Assume that $P(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$ and that $r(A - \lambda_1 I) = 2$. Then, the J. f. of matrix A is

$$(6) \quad J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}.$$

Proof. Denote $B = (A - \lambda_1 I)(A - \lambda_2 I)$ and suppose that $(A - \lambda_1 I)B \neq 0$. Then, there exists a column v_3 of B , for which $(A - \lambda_1 I)v_3 = v_2 \neq \theta$. Because $(A - \lambda_1 I)v_2 = (A - \lambda_1 I)^2 v_3 = \theta$, it follows that $Av_2 = \lambda_1 v_2$. Taking v_1 such that $Av_1 = \lambda_1 v_1$ and that $\{v_1, v_2\}$ be l. i. and $v_4 \neq \theta$ such that $Av_4 = \lambda_2 v_4$, we obtain the vectors v_i , $i = \overline{1, 4}$ satisfying the relations

$$(7) \quad Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2, \quad Av_3 = \lambda_1 v_3 + v_2, \quad Av_4 = \lambda_2 v_4.$$

If we put $H = \text{col}[v_1, v_2, v_3, v_4]$, we have $AH = HJ$, where J is given by (6) and the conclusion holds. For the second part of the proof, we suppose that $(A - \lambda_1 I)B = 0$. Let v be a column of $A - \lambda_2 I$, such that $(A - \lambda_1 I)v \neq 0$. Denoting $v_3 = v$ and $v_2 = (A - \lambda_1 I)v_3$, we have $(A - \lambda_1 I)v_2 = \theta$, if make use of the relation $(A - \lambda_1 I)B = 0$. We have thus obtained two nonzero vectors v_2 and v_3 such that $Av_2 = \lambda_1 v_2$, $Av_3 = \lambda_1 v_3 + v_2$. If we choose $v_1 \neq \theta$ such that $Av_1 = \lambda_1 v_1$ and $\{v_1, v_2\}$ be l. i., we again arrive to the relations (7) and the proof is complete.

Proposition 5. Assume that $P(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$ and that $r(A - \lambda_1 I) = 3$. Then, the J. f. of matrix A is given by

$$(8) \quad J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}.$$

Proof. Making use of Lemma 2, it follows that $(A - \lambda_1 I)^2(A - \lambda_2 I) \neq \theta$ and hence there exists a column v of $A - \lambda_2 I$ such that $(A - \lambda_1 I)^2 v \neq \theta$. Obviously $(A - \lambda_1 I)v \neq \theta$ and if we denote $v_3 = v$, $v_2 = (A - \lambda_1 I)v_3$, $v_1 = (A - \lambda_1 I)v_2$ we obtain that $v_1 \neq \theta$ and that $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_1 v_2 + v_1$, $Av_3 = \lambda_1 v_3 + v_2$. If $v_4 \neq \theta$ satisfies $Av_4 = \lambda_2 v_4$, we have four vectors v_i which are l. i. (exercice) and the relation $AH = HJ$, where J is given by (8) and $H = \text{col}[v_1, v_2, v_3, v_4]$. This ends the proof.

Remark 2. In a concrete case, we firstly find the vector $v_1 \neq 0$ from the system $(A - \lambda_1 I)v_2 = v_1$ and we obtain the vector v_2 and finally we solve the system $(A - \lambda_1 I)v_3 = v_2$, which gives the vector v_3 . The compatibility of these systems is established by the preceding Proposition 5. Hence, the most simple technique (linear systems) is sufficient to obtain the J. f. and the matrix H which leads to the J. f. in this case, like as in all possible cases.

Finally, let us consider the case $P(\lambda) = (\lambda - \lambda_1)^1$. If $r(A - \lambda_1 I) = 0$, then $A = \lambda_1 I$ has the J. f. and the problem is not interesting.

Proposition 6. Assume that $P(\lambda) = (\lambda - \lambda_1)^4$ and that $r(A - \lambda_1 I) = 1$. Then, the J. f. of matrix A is

$$(9) \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \text{where} \quad J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

Proof. Suppose that $(A - \lambda_1 I)^3 \neq 0$ and denote by v a column of $A - \lambda_1 I$, for which $v_3 = (A - \lambda_1 I)^2 v \neq 0$. We have $(A - \lambda_1 I)v_3 = (A - \lambda_1 I)^3 v = 0$, because $(A - \lambda_1 I)^4 = 0$. Hence v_3 is an eigenvector of A . If we put $v_4 = (A - \lambda_1 I)v$, it follows that $(A - \lambda_1 I)v_4 = v_3$. Let v_1 and v_2 be two eigenvectors of the matrix A such that $\{v_1, v_2, v_3\}$ be l. i. Then, we have

$$(10) \quad Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2, \quad Av_3 = \lambda_1 v_3, \quad Av_4 = \lambda_1 v_4 + v_3$$

and the conclusion follows. Now, suppose that $(A - \lambda_1 I)^3 = 0$, but $(A - \lambda_1 I)^2 \neq 0$. We take a column w of $A - \lambda_1 I$, for which $v = (A - \lambda_1 I)w \neq 0$. We remark that $(A - \lambda_1 I)v = 0$, hence v is an eigenvector of A . If we put $v_3 = v$, $v_4 = w$ and denote by v_1, v_2 two eigenvectors of A such that $\{v_1, v_2, v_3\}$ be l. i., we again obtain the relations (10) and the conclusion also follows, because it may be easily proved that $v_i, i = \overline{1, 4}$ are l. i. (exercice). Finally, suppose that $(A - \lambda_1 I)^2 = 0$ and take a vector w , such that $v = (A - \lambda_1 I)w \neq 0$; this fact is possible, because $A - \lambda_1 I \neq 0$. If we denote $v_3 = v$, $v_4 = w$ and v_1, v_2 are chosen as in the preceding situations, we again arrive to the relations (10) and this ends the proof.

Proposition 7. Assume that $P(\lambda) = (\lambda - \lambda_1)^4$ and that $r(A - \lambda_1 I) = 2$. Then, the J. f. of matrix A is

$$(11) \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \text{where} \quad J_1 = J_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

Proof. Suppose that $(A - \lambda_1 I)^3 \neq 0$ and remark that $A - \lambda_1 I$ has two l. i. columns v and w . Then, the vectors $(A - \lambda_1 I)^2 v$ and $(A - \lambda_1 I)^2 w$ can not be both equal to zero. Under the supplementary assumption that they are l. i., we put $v_1 = (A - \lambda_1 I)^2 v$, $v_2 = (A - \lambda_1 I)v$, $v_3 = (A - \lambda_1 I)^2 w$, $v_4 = (A - \lambda_1 I)w$ and we have

$$(12) \quad Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2 + v_1, \quad Av_3 = \lambda_1 v_3, \quad Av_4 = \lambda_1 v_4 + v_3,$$

the vectors $v_i, i = \overline{1, 4}$ being l. i. (exercice). Putting $H = \text{col}[v_1, v_2, v_3, v_4]$, we have $H^{-1}AH = J$, where J is given by (11). However, the conclusion holds without the supplementary hypothesis on the vectors $(A - \lambda_1 I)^2 v$, $(A - \lambda_1 I)^2 w$, but we omit this part of the proof.

Now, suppose that $(A - \lambda_1 I)^3 = 0$ and $(A - \lambda_1 I)^2 \neq 0$. The vectors v and w having the same meaning as above, making the assumption that $(A - \lambda_1 I)v$

and $(A - \lambda_1 I)w$ are l. i. and putting $v_1 = (A - \lambda_1 I)v$, $v_2 = v$, $v_3 = (A - \lambda_1 I)w$, $v_4 = w$ we again obtain the relations (12) and the desired conclusion. Finally, we suppose that $(A - \lambda_1 I)^2 = 0$ and we mention the assumption that v and w are two l. i. columns of the matrix $A - \lambda_1 I$. Thus, we have $(A - \lambda_1 I)v = 0$, $(A - \lambda_1 I)w = 0$, i. e. v and w are eigenvectors of A . We denote $v_1 = v$, $v_3 = w$ and we remark that the systems $(A - \lambda_1 I)v_2 = v_1$, $(A - \lambda_1 I)v_4 = v_3$ have solutions v_2 and respectively v_4 . For example, if v_1 is the first column of $A - \lambda_1 I$, it follows $v_2^T = [1, 0, 0, 0]$; if v_3 is the third column of $A - \lambda_1 I$, it follows $v_4^T = [0, 0, 1, 0]$. We also have the relations (12), in which the vectors v_i , $i = \overline{1, 4}$ are l. i. (exercice) and this ends the proof.

Remark 3. The proof of the preceding Proposition 7 seems to be relatively complicated. But in a concrete situation, nothing is more simple than to find the vectors v_i , $i = \overline{1, 4}$. Namely, (first we) solve the system $(A - \lambda_1 I)v = 0$ and we obtain two l. i. eigenvectors v_1 and v_3 (we remember that $r(A - \lambda_1 I) = 2$). Then, we solve the (compatible) systems $(A - \lambda_1 I)v = v_1$ and $(A - \lambda_1 I)v = v_3$ and we obtain the solution v_2 , respectively v_4 . The relations (12) are fulfilled and hence the J. f. of the matrix A is obtained in the most elementary possible way.

The last case which we may encounter is described in the following

Proposition 8. Assume that $P(\lambda) = (\lambda - \lambda_1)^4$ and that $r(A - \lambda_1 I) = 3$. Then, the J. f. of matrix A is

$$(13) \quad J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}.$$

Proof. We remark that $r(A - \lambda_1 I)^3 \geq 1$ and hence $(A - \lambda_1 I)^3 \neq 0$. Let v , w and u be three l. i. columns of the matrix $A - \lambda_1 I$. Between the vectors

$$(14) \quad (A - \lambda_1 I)^2 v, \quad (A - \lambda_1 I)^2 w, \quad (A - \lambda_1 I)^2 u$$

there exists at least one which is different from zero, otherwise we would have $(A - \lambda_1 I)^3 = 0$, which is impossible. Supposing that $(A - \lambda_1 I)^2 v \neq 0$, we put

$$(15) \quad v_1 = (A - \lambda_1 I)^2 v, \quad v_2 = (A - \lambda_1 I)v, \quad v_3 = v$$

and we observe that the linear system $(A - \lambda_1 I)v_4 = v_3$ is compatible because v_3 is a column of $A - \lambda_1 I$. Hence we have obtained four vectors v_i , $i = \overline{1, 4}$ which are l. i. (exercice). Having

$$(16) \quad Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2 + v_1, \quad Av_3 = \lambda_1 v_3 + v_2, \quad Av_4 = \lambda_1 v_4 + v_3$$

and putting $H = \text{col}[v_1, v_2, v_3, v_4]$, we conclude that $AH = HJ$, where J is given by (13) and this ends the proof.

Remark 4. We point out that the case described in Proposition 8 ($P(\lambda) = (\lambda - \lambda_1)^4$, $r(A - \lambda_1 I) = 3$) is simpler than other cases, although the matrix A has only one eigenvector (neglecting the colinear vectors). The practical rule applied in this case in the following: first we solve the system $(A - \lambda_1 I)v = 0$ and we fix a solution $v_1 = 0$; with v_1 fixed, the last three systems in (16) are compatible and we obtain the J. f. of A , making no use of other operations except solving linear systems of equations.

Remark 5. Similar considerations may be made about the matrix of the second and third order.

Received February 7, 1992

*Polytechnic Institute, Jassy,
Department of Mathematics*

REFERENCES

1. Гантмахер Ф. Р., *Теория матриц*. Москва, 1953, pp. 61, 74.

O METODĂ DIRECTĂ PENTRU FORMA JORDAN A MATRICELOR DE ORDINUL PATRU

(Rezumat)

Se prezintă cea mai elementară metodă de găsire a formei canonice Jordan, care constă în esență din rezolvarea unor sisteme liniare de patru ecuații cu patru necunoscute și nu presupune cunoștințe de algebră superioară.

SOME ORDERED GROUPS ON SOME CARTESIAN PRODUCTS (II)

BY

GH. COSTOVICI

1. By means of bilinear mappings defined on some cartesian products of partially ordered (p. o.) groups, we obtain some new p. o. groups and give more examples.

2. If G_1 , G_2 and G are multiplicative groups, then the mapping $f: G_1 \times G_2 \rightarrow G$ is bilinear $\Leftrightarrow$ are fulfilled

$$(1) \quad \forall x, y \in G_1 \text{ and } \forall z \in G_2, \quad f(xy, z) = f(x, z)f(y, z) \text{ and}$$

$$(2) \quad \forall x \in G_1 \text{ and } \forall y, z \in G_2, \quad f(x, yz) = f(x, y)f(x, z).$$

If $f: G_1 \times G_2 \rightarrow G$ is bilinear then we have

$$(3) \quad \forall x \in G_1 \text{ and } \forall y \in G_2, \quad f(x, 1) = 1 \text{ and } f(1, y) = 1.$$

Indeed $f(1, y) = f(11, y) = f(1, y)f(1, y)$.

Proposition (P). Let $f_i: G_i \times G_i \rightarrow G$, $i = \overline{1, p}$ be bilinear mappings and G commutative. One considers $G_1 \times \dots \times G_p$ group by the usual operation $(x_1, \dots, x_p)(y_1, \dots, y_p) = (x_1y_1, \dots, x_py_p)$. Then $f: (G_1 \times \dots \times G_p) \times (G_1 \times \dots \times G_p) \rightarrow G$ with $f((a_1, \dots, a_p), (b_1, \dots, b_p)) = f_1(a_1, b_1) \dots f_p(a_p, b_p)$ is bilinear.

Proof. $f((a_1, \dots, a_p)(c_1, \dots, c_p), (b_1, \dots, b_p)) = f((a_1c_1, \dots, a_pc_p), (b_1, \dots, b_p)) = f_1(a_1c_1, b_1) \dots f_p(a_pc_p, b_p) = f_1(a_1, b_1)f_1(c_1, b_1) \dots f_p(a_p, b_p)f_p(c_p, b_p) = f((a_1, \dots, a_p), (b_1, \dots, b_p))f((c_1, \dots, c_p), (b_1, \dots, b_p))$ and similarly follows (2).

Remark. One can prove a similar proposition for f_i being q -linear.

Theorem (T). I. If $f, g: G_1 \times G_2 \rightarrow G$ are bilinear mappings and G is commutative, then $H = G_1 \times G_2 \times G$ becomes group by the operation

$$(a, b, c) \circ (x, y, z) = (ax, by, czf(a, y)g(x, b)).$$

II. If besides the hypotheses of I, we suppose that G_1, G_2 and G are p. o. groups, then $(H, \circ)$ becomes p. o. group by the lexicographic order that is with positive cone $P = \{(x, y, z) \in H; x > 1 \text{ or } (x = 1 \text{ and } y > 1) \text{ or } (x = 1, y = 1 \text{ and } z \geq 1)\}$.

Proof of I. $[(a, b, c) \circ (x, y, z)] \circ (u, v, w) = (ax, by, czf(a, y)g(x, b)) \circ (u, v, w) = (axu, byv, czf(a, y)g(x, b)wf(ax, v)g(u, by)) = (axu, byv, czf(a, y)g(x, b)wf(ax, v)g(u, by)) = (axu, byv, czf(a, y)g(x, b)wf(ax, v)g(u, by))$.

$(a, b, c) \circ [(x, y, z) \circ (u, v, w)] = (a, b, c) \circ (xu, yv, zwf(x, v)g(u, y)) = (axu, byv, czwf(x, v)g(u, y)f(a, yv)g(xu, b)) = (axu, byv, czwf(x, v)g(u, y)f(a, y)f(a, v)g(x, b)g(u, b)).$

The group G being commutative, the operation $\circ$ is associative. $\exists(1, 1, 1) \in H$ so that $\forall(x, y, z) \in H$ we have $(1, 1, 1) \circ (x, y, z) = (1x, 1y, 1zf(1, y)g(x, 1)) = (x, y, z)$ by (3).

If $(x, y, z) \in H$, $\exists(x^{-1}, y^{-1}, z^{-1}f(x, y)g(x, y)) \in H$ so that $(x^{-1}, y^{-1}, z^{-1}f(x, y)g(x, y)) \circ (x, y, z) = (1, 1, z^{-1}f(x, y)g(x, y)zf(x^{-1}, y)g(x, y^{-1})) = (1, 1, 1)$ by the commutativity of G and by (3).

Proof of II. $P = \{(x, y, z) \in H; x > 1 \text{ or } (x=1 \text{ and } y > 1) \text{ or } (x=1, y=1 \text{ and } z \geq 1)\}$. We show that $P \cap P^{-1} = \{(1, 1, 1)\}$. Let $(x, y, z) \in P \cap P^{-1}$. Then $(x, y, z) \in P$ and $(x, y, z) = (p_1, p_2, p_3)^{-1} = (p_1^{-1}, p_2^{-1}, p_3^{-1}f(p_1, p_2)g(p_1, p_2))$ with $(p_1, p_2, p_3) \in P$. $x > 1$ and $x = p_1^{-1} \leq 1$, impossibly; $x=1=p_1^{-1}$, $y > 1$ and $y = p_2^{-1} \leq 1$, impossibly; $x=1=p_1^{-1}$, $y=1=p_2^{-1}$ and $z \geq 1 \Rightarrow z = p_3^{-1}f(p_1, p_2)g(p_1, p_2) = p_3^{-1} \leq 1 \Rightarrow x=1, y=1, z=1$.

We show that $P \circ P \subseteq P$. Let $(x, y, z) \in P$ be $\Leftrightarrow x > 1$ or $(x=1 \text{ and } y > 1)$ or $(x=1, y=1 \text{ and } z \geq 1)$. Let $(u, v, w) \in P$ be $\Leftrightarrow u > 1$ or $(u=1 \text{ and } v > 1)$ or $(u=1, v=1 \text{ and } w \geq 1)$. We have $t = (x, y, z) \circ (u, v, w) = (xu, yv, zwf(x, v)g(u, y))$.

$x > 1$ and $u > 1 \Rightarrow xu > 1 \Rightarrow t \in P$; $x > 1$, $u=1$ and $v > 1 \Rightarrow xu > 1 \Rightarrow t \in P$; $x=1, y > 1, u > 1 \Rightarrow xu > 1 \Rightarrow t \in P$; $x=1, y > 1, u=1$ and $v > 1 \Rightarrow xu=1$ and $yv > 1 \Rightarrow t \in P$; $x=1, y > 1, u=1, v=1, w \geq 1 \Rightarrow xu=1$ and $yv > 1 \Rightarrow t \in P$; $x=1, y=1, z \geq 1, u > 1 \Rightarrow xu > 1 \Rightarrow t \in P$; $x=1, y=1, z \geq 1, u=1, v > 1 \Rightarrow xu=1$ and $yv > 1 \Rightarrow t \in P$; $x=1, y=1, z \geq 1, u=1, v=1, w \geq 1 \Rightarrow xu=1, yv=1$ and $zw \geq 1 \Rightarrow t \in P$.

We show that $\forall(x, y, z) \in H$ we have $(x, y, z) \circ P \circ (x, y, z)^{-1} \subseteq P$. Let $(p_1, p_2, p_3) \in P$ be $\Leftrightarrow p_1 > 1$ or $(p_1=1 \text{ and } p_2 > 1)$ or $(p_1=1, p_2=1 \text{ and } p_3 \geq 1)$. We have $t = (x, y, z) \circ (p_1, p_2, p_3) \circ (x^{-1}, y^{-1}, z^{-1}f(x, y)g(x, y)) = (xp_1, yp_2, zp_3f(x, p_2)g(p_1, y)) \circ (x^{-1}, y^{-1}, z^{-1}f(x, y)g(x, y)) = (xp_1x^{-1}, yp_2y^{-1}, zp_3f(x, p_2)g(p_1, y)z^{-1}f(x, y)g(x, y)f(xp_1, y^{-1})g(x^{-1}, yp_2)) = (xp_1x^{-1}, yp_2y^{-1}, p_3f(x, p_2)g(p_1, y)f(x, y)g(x, y)f(x, y^{-1})f(p_1, y^{-1})g(x^{-1}, y)g(x^{-1}, p_2)) = (xp_1x^{-1}, yp_2y^{-1}, p_3f(x, p_2)g(p_1, y)f(p_1, y^{-1})g(x^{-1}, p_2)).$

$p_1 > 1 \Rightarrow xp_1x^{-1} > 1 \Rightarrow t \in P$; $p_1=1$ and $p_2 > 1 \Rightarrow xp_1x^{-1}=1$ and $yp_2y^{-1} > 1 \Rightarrow t \in P$; $p_1=1, p_2=1$ and $p_3 \geq 1 \Rightarrow xp_1x^{-1}=1, yp_2y^{-1}=1$ and $p_3f(x, p_2)g(p_1, y)f(p_1, y^{-1})g(x^{-1}, p_2) = p_3 \geq 1 \Rightarrow t \in P$.

Corollary 1 (C_1). I. If $f: G_1 \times G_2 \rightarrow G$ is bilinear mapping and G is commutative, then $H = G_1 \times G_2 \times G$ becomes group by the every operation „ $\circ_i$ “, $i=1, 3$, where $(a, b, c) \circ_i (x, y, z) = (ax, by, czf_i)$, $i=1, 3$, and where $t_1=f(a, y)$, $t_2=f(x, b)$, $t_3=f(a, y)f(x, b)=t_1t_2$.

II. If besides the hypotheses of I, the groups G_1, G_2 and G are p. o. groups, then $(H, \circ_i)$ $i=1, 3$, becomes p. o. group by the lexicographic order that is with the cone from (T).

The proof for $\circ_1$ follows from (T) if we take $g(x, y)=1 \forall(x, y) \in G_1 \times G_2$, the proof for $\circ_2$ follows from (T) if we take $f(x, y)=1 \forall(x, y) \in G_1 \times G_2$, and the proof for $\circ_3$ follows from (T) if $\forall(x, y) \in G_1 \times G_2, f(x, y)=g(x, y)$. The following Corollary 2 is a special case of (C_1) via (P).

Corollary 2 (C_2). I. If $f_i: G_i \times G_i \rightarrow G$ $i=1, p$, are bilinear mappings and G is commutative, then $H = (G_1 \times \dots \times G_p) \times (G_1 \times \dots \times G_p) \times G$ becomes group by the operation „ $\circ_i$ “, $i=1, 3$, where $((a_1, \dots, a_p), (b_1, \dots, b_p), c_1) \circ_i ((x_1, \dots, x_p),$

$(y_1, \dots, y_p), c_2) = ((a_1x_1, \dots, a_px_p), (b_1y_1, \dots, b_py_p), c_1c_2t_1)$ and where $t_1 = f_1(a_1, y_1) \dots f_p(a_p, y_p)$, $t_2 = f_1(x_1, b_1) \dots f_p(x_p, b_p)$, $t_3 = t_1t_2$.

II. If besides the hypotheses of I (here), the groups G_i , $i = \overline{1, p}$, and G are p. o. groups, then, considering $G_1 \times \dots \times G_p$ p. o. group by the usual operation and by the lexicographic order, the group $(H, \circ_i)$, $i = \overline{1, 3}$, becomes p. o. group by the lexicographic order that is with the cone $P = \{((x_1, \dots, x_p), (y_1, \dots, y_p), g) \in H; (x_1, \dots, x_p) > (1, \dots, 1) \text{ or } ((x_1, \dots, x_p) = (1, \dots, 1) \text{ and } (y_1, \dots, y_p) > (1, \dots, 1)) \text{ or } ((x_1, \dots, x_p) = (1, \dots, 1), (y_1, \dots, y_p) = (1, \dots, 1) \text{ and } g \geq 1)\}$.

Examples. Let C be the additive group of complex numbers (or complex rational numbers). One verifies that $f_i: C \times C \rightarrow R$, $i = \overline{1, 8}$ where

$$f_1((a, b)(x, y)) = \operatorname{Re}((a, b)(x, y)) = ax - by,$$

$$f_2((a, b), (x, y)) = \operatorname{Re}((a, b)(x, -y)) = ax + by,$$

$$f_3((a, b)(x, y)) = \operatorname{Re}((a, -b)(x, y)) = ax + by = f_2((a, b), (x, y)),$$

$$f_4((a, b), (x, y)) = \operatorname{Re}((a, -b)(x, -y)) = ax - by = f_1((a, b), (x, y)),$$

$$f_5((a, b), (x, y)) = \operatorname{Im}((a, b)(x, y)) = ay + bx,$$

$$f_6((a, b), (x, y)) = \operatorname{Im}((a, b)(x, -y)) = -ay + bx,$$

$$f_7((a, b), (x, y)) = \operatorname{Im}((a, -b)(x, y)) = ay - bx,$$

$$f_8((a, b), (x, y)) = \operatorname{Im}((a, -b)(x, -y)) = -ay - bx, \text{ are bilinear.}$$

Then, by (C_2) , I, and by means of these six bilinear mappings, $H = C^n \times C^n \times R$ becomes group by the operation $\circ_1(\circ_2)(\circ_3)$ in $6^n(6^n) = (6^{2n})$ ways; respectively. In ([2], p. 16) one considers C partially ordered in five ways. Then, by (C_2) , I and II, $H = C^n \times C^n \times R$ becomes p. o. group by $\circ_1(\circ_2)(\circ_3)$ in $5^{2n} \times 6^n(5^{2n} \times 6^n)(5^{2n} \times 6^{2n})$ ways, respectively. Similarly can be considered complex rational numbers and Q .

In [1] there are several examples related to (C_1) .

Received February 25, 1991

Polytechnic Institute, Jassy,
Department of Mathematics

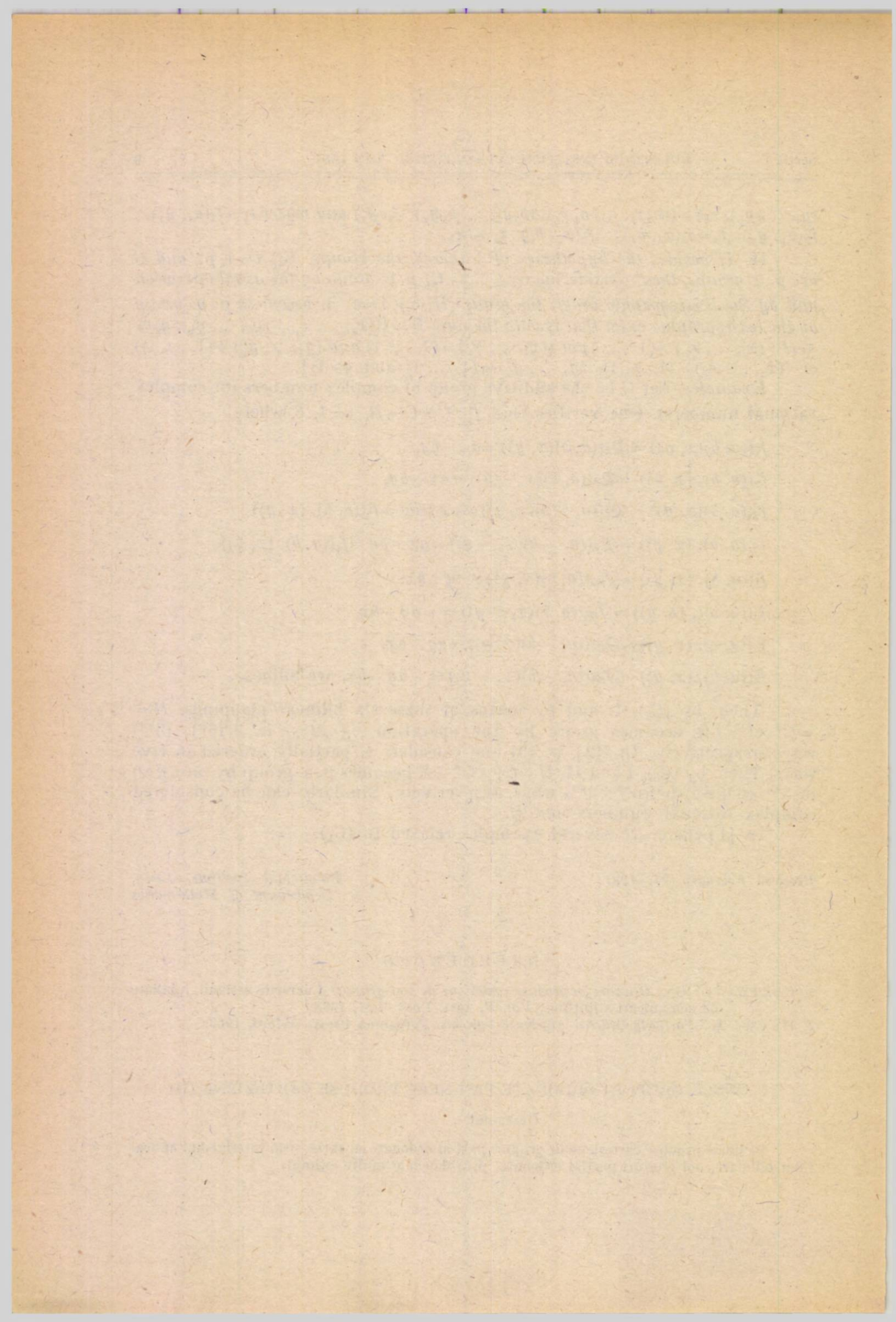
REFERENCES

1. Costovici Gh., *Grupuri pe produse carteziene de trei grupuri*. Lucrările sesiunii jubiliare de comunicări științifice, Vol. V, Inst. Polit. Iași, 1988.
2. Fuchs L., *Partially Ordered Algebraic Systems*. Pergamon Press, Oxford, 1963.

UNELE GRUPURI ORDONATE PE UNELE PRODUSE CARTEZIENE (II)

(Rezumat)

Pe unele produse carteziene de grupuri parțial ordonate se obțin, prin intermediul aplicațiilor biliniare, noi grupuri parțial ordonate, și se dau mai multe exemple.



ON GEOMETRY OF VERTICAL VECTOR BUNDLE

BY

P. TAMIA-DIMOPOULOU

1. Introduction. Aurel Bejancu in [3] has started a study of submanifolds of a manifold whose vertical bundle is endowed with a pseudo-Riemannian structure. It is very important to note that Finsler spaces and Lagrange spaces are examples of manifolds such that each of their vertical bundles has a pseudo-Riemannian structure. In this way, it has been obtained, as a particular case, a coordinate free theory for Finsler subspaces and Lagrange subspaces. A vectorial connection on a manifold whose vertical bundle is endowed with a pseudo-Riemannian structure defines two general vectorial Finsler connections [3]. The vectorial connection and the two general vectorial Finsler connections induce new connections on a submanifold of a such manifold [3], [5], [6]. The purpose of this paper is to investigate interrelations of the induced vectorial connection and the induced general vectorial Finsler connections on the submanifolds.

2. Preliminaries. Let $\tilde{M}$ be a real m -dimensional differentiable manifold and $T\tilde{M}$ be the tangent bundle to $\tilde{M}$. Denote by $\pi: T\tilde{M} \rightarrow \tilde{M}$ the canonical projection of $T\tilde{M}$ to $\tilde{M}$ and by $d\pi: TT\tilde{M} \rightarrow T\tilde{M}$ its differential. The vertical vector bundle on $T\tilde{M}$ is defined as the kernel of $d\pi$ and it is denoted by $VT\tilde{M}$. The vertical vector bundle on $T\tilde{M}$ is an integrable distribution on $T\tilde{M}$, whose fibers are m -dimensional at each $z \in T\tilde{M}$. A complementary distribution $HT\tilde{M}$ to $VT\tilde{M}$ to $TT\tilde{M}$ is called a non linear connection on $T\tilde{M}$. In the sequel the Latin indices i, j, k , run from 1, 2, ... to m and Greek indices run from 1 to n .

If (U, φ, R^{2n}) and (U', φ', R^{2n}) are two charts on $T\tilde{M}$ then for each $z \in U \cap U' \neq \emptyset$ we denote by $(x^1, \dots, x^n, y^1, \dots, y^n)$ and $(x^{1'}, \dots, x^{n'}, y^{1'}, \dots, y^{n'})$ its coordinates in these charts. Then the non-linear connection $HT\tilde{M}$ is given by n^2 differentiable functions N_j^i on each U satisfying

$$N_{j'}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \left(\frac{\partial x^j}{\partial x^{j'}} N_j^i + \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}} y^{k'} \right)$$

at all points of $U \cap U'$, where $N_{j'}^{i'}$ are the corresponding functions on U' .

Let $\{\partial/\partial x^i, \partial/\partial y^i\}$ be the natural local field of frames on $T\tilde{M}$. Then $\{\delta/\delta x^i\}$ given by $\delta/\delta x^i = \partial/\partial x^i - N_i^j \partial/\partial y^j$ is a local field of frames on the

distribution $HT\tilde{M}$, $\{\partial/\partial y^i\}$ is a local field of frames on $VT\tilde{M}$ and $\{\delta/\delta x^i, \partial/\partial y^i\}$ is a local field of frames on $T\tilde{M}$ adapted to the decomposition $TT\tilde{M} = HT\tilde{M} \oplus VT\tilde{M}$.

Now let $\pi_E: E \rightarrow T\tilde{M}$ be a Finsler vector bundle on $\tilde{M}$ with standard fibre R^p and $\Gamma(E)$ be the $C(T\tilde{M})$ -module of all differentiable cross sections of E , where $C(T\tilde{M})$ is the algebra of all differentiable real functions on $T\tilde{M}$. We keep the same notation Γ for the differentiable cross sections of any other vector bundle. Also let $P: E \rightarrow E$ be a vector bundle morphism. A *general connection* or an *Otsuki connection* with respect to P is a mapping $\nabla: \Gamma(TT\tilde{M}) \times \Gamma(E) \rightarrow \Gamma(E)$ satisfying

$$(2.1) \quad \begin{aligned} \nabla_{fX+Y}S &= f\nabla_X S + \nabla_Y S, \\ \nabla_X(fS+S') &= (Xf)PS + f\nabla_X S + \nabla_X S', \end{aligned}$$

for any $S, S' \in \Gamma(E)$, $X, Y \in \Gamma(TT\tilde{M})$ and $f \in C(T\tilde{M})$.

A general Finsler connection on the Finsler vector bundle E is the triplet (N, P, ∇) , which we will denote *GFC* for brevity [4]. If in particular P is the identity morphism I_E on E then ∇ is just a linear connection on E , and we obtain the concept of vectorial Finsler connection on E [2]. The curvature tensor R of ∇ is given by [1].

$$(2.2) \quad \begin{aligned} R(X, Y)V &= \nabla_X \nabla_Y PV - \nabla_Y \nabla_X PV - P(\nabla_{[X, Y]}PV) - \\ &\quad - (\nabla_X I_E) \nabla_Y V - (\nabla_Y I_E) \nabla_X V, \end{aligned}$$

for any $X, Y \in \Gamma(TT\tilde{M})$ and $V \in \Gamma(E)$. We see that $R(X, Y)$ is an endomorphism of $\Gamma(E)$. If $E = TT\tilde{M}$, then the torsion tensor of ∇ is defined by

$$(2.3) \quad T(X, Y) = \nabla_X Y - \nabla_Y X - P[X, Y].$$

3. Pseudo-Riemannian Structures on the Vertical Vector Bundle to a Manifold. Let $\tilde{M}$ be a real m -dimensional differentiable manifold and $TT\tilde{M} = HT\tilde{M} \oplus VT\tilde{M}$ be the tangent bundle to $T\tilde{M}$. A map

$$g: \Gamma(VT\tilde{M}) \times \Gamma(VT\tilde{M}) \rightarrow C(T\tilde{M})$$

is called a pseudo-Riemannian metric on the vertical bundle if it is symmetric, $C(T\tilde{M})$ -bilinear and non degenerate on each fibre of $VT\tilde{M}$ [3].

We denote by $\tilde{v}$ and $\tilde{h}$ the projection morphisms of $TT\tilde{M}$ to $VT\tilde{M}$ and $HT\tilde{M}$ respectively. Then an automorphism $\tilde{P}: TT\tilde{M} \rightarrow TT\tilde{M}$ can be defined by

$$\tilde{P}(X) = X_H^i(x, y) \frac{\partial}{\partial y^i} + X_V^i(x, y) \frac{\delta}{\delta x^i} \quad \text{for} \quad X = X_H^i(x, y) \frac{\delta}{\delta x^i} + X_V^i(x, y) \frac{\partial}{\partial y^i}.$$

It is easy to see that the definition of $\tilde{P}$ is independent of the coordinate systems on $T\tilde{M}$ and $\tilde{P} = I_{TT\tilde{M}}^2, \tilde{P}\tilde{h} = \tilde{v}\tilde{P}$. Moreover, $\tilde{P}$ is carrying $HT\tilde{M}$ on $VT\tilde{M}$ and viceversa (see [3]).

Let $\tilde{\nabla}$ be a linear connection on $VT\tilde{M}$. The linear connection $\tilde{\nabla}$ defines two Otsuki connections [3] by

$$(3.1) \quad \tilde{D}_X = \tilde{\nabla}_X \tilde{V},$$

$$(3.2) \quad \tilde{D}'_X = \tilde{P}\tilde{\nabla}_X\tilde{P}\tilde{h} = \tilde{P}\tilde{D}_X\tilde{P} \quad \text{for any } X \in \Gamma(TT\tilde{M}),$$

with respect to $\tilde{v}$ and $\tilde{h}$ respectively.

We call the triplet $(HT\tilde{M}, \tilde{v}, \tilde{D})$ *first general vectorial Finsler connection* and the triplet $(HT\tilde{M}, \tilde{h}, \tilde{D}')$ *second general vectorial Finsler connection* induced by $\tilde{\nabla}$.

We will say that $\tilde{\nabla}$ is a metric-connection on $VT\tilde{M}$ with respect to pseudo-Riemannian metric g , if $\tilde{\nabla}g=0$, that is

$$(\tilde{\nabla}_X g)(Y, Z) = X(g(Y, Z)) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) = 0$$

for any $X \in \Gamma(TT\tilde{M})$ and $Y, Z \in \Gamma(VT\tilde{M})$.

Theorem 3.1 [3]. Suppose g is a pseudo-Riemannian metric on $VT\tilde{M}$ and $HT\tilde{M}$ is a non-linear connection on $T\tilde{M}$. Then there exists a unique metric connection $\tilde{\nabla}$ on $VT\tilde{M}$ satisfying

$$\tilde{\nabla}_Y Z - \tilde{\nabla}_Z Y - [Y, Z] = 0 \quad \text{for any } Y, Z \in \Gamma(VT\tilde{M})$$

and

$$\tilde{\nabla}_U \tilde{P}V - \tilde{\nabla}_V \tilde{P}U - \tilde{P}\tilde{h}([U, V]) = 0 \quad \text{for any } U, V \in \Gamma(HT\tilde{M}).$$

The connection $\tilde{\nabla}$ which is uniquely defined by Theorem 3.1 is called *(v)-Levi-Civita connection* on $VT\tilde{M}$ with respect to $HT\tilde{M}$.

4. Submanifolds of a Manifold whose Vertical Bundle has a Pseudo-Riemannian Structure. Suppose M is a real n -dimensional submanifold of a m -dimensional differentiable manifold whose vertical bundle $VT\tilde{M}$ is endowed with a pseudo-Riemannian metric. Let the submanifold of $\tilde{M}$ be given locally by equations

$$x^i = x^i(u^1, \dots, u^n), \quad i=1, \dots, m \quad \text{and} \quad \text{rank} \left[\frac{\partial x^i}{\partial u^\alpha} \right] = n < m.$$

Denote by i the immersion of M in $\tilde{M}$ and by di its differential. We will identify M with its image $i(M)$ on $\tilde{M}$ and $di(TM)$ with TM . Then locally a point (u^α, v^α) of TM is identified with $(x^i(u), y^i(u, v))$ of $T\tilde{M}$.

Moreover we have $y^i = B_\alpha^i v^\alpha$, where $B_\alpha^i = \partial x^i / \partial u^\alpha$, $i=1, \dots, m$, $\alpha=1, \dots, n$,

$$(4.1) \quad \frac{\partial}{\partial v^\alpha} = B_\alpha^i \frac{\partial}{\partial y^i},$$

$$(4.2) \quad \frac{\partial}{\partial u^\alpha} = B_\alpha^i \frac{\partial}{\partial x^i} + B_{\alpha\beta}^i \frac{\partial}{\partial y^i}, \quad \text{where} \quad B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} v^\beta.$$

From (4.1) it follows that the vertical bundle to M denoted by VTM is a vector subbundle of $VT\tilde{M}$.

We denote by $VTM^\perp$ the v -normal bundle to M in $\tilde{M}$ which is the complementary orthogonal subbundle to VTM in $VT\tilde{M}$. That is $VT\tilde{M} = VTM \oplus VTM^\perp$. Of course, the dimension of each fiber of $VTM^\perp$ is $m-n$, which is equal to the codimension of M in $\tilde{M}$. The pseudo-Riemannian metric g on the vertical bundle $VT\tilde{M}$ induces in a natural way a symmetric $C(TM)$ -bilinear mapping $g_v: \Gamma(VTM) \times \Gamma(VTM) \rightarrow C(TM)$ and a symmetric $C(TM)$ -bilinear mapping $g_\perp: \Gamma(VTM^\perp) \times \Gamma(VTM^\perp) \rightarrow C(TM)$. In the sequel we suppose g_v and $g_\perp$ are also nondegenerate, that is, they are induced pseudo-Riemannian metrics on VTM and $VTM^\perp$ respectively.

Theorem 4.1 [3]. *Let $HT\tilde{M}$ be a non-linear connection on $T\tilde{M}$. Then there exists a unique non-linear connection HTM on TM such that HTM is a vector subbundle of $HT\tilde{M} \oplus VTM^\perp$ (direct sum).*

The unique non-linear connection HTM is called *induced non-linear connection on TM* .

Let $\tilde{\nabla}$ be a linear connection on $VT\tilde{M}$. The connection induces a linear connection ∇ on VTM and a linear connection $\nabla^\perp$ on $VTM^\perp$, which are defined by

$$(4.3) \quad \tilde{\nabla}_X Y = \nabla_X Y + B(X, Y): (v)\text{-Gauss formula } \forall X \in \Gamma(TTM), Y \in \Gamma(VTM),$$

$$(4.4) \quad \tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V: (v)\text{-Weingarten formula } \forall X \in \Gamma(TTM), \\ V \in \Gamma(VTM^\perp),$$

where $B: \Gamma(TTM) \times \Gamma(VTM) \rightarrow \Gamma(VTM^\perp)$ is a $C(TM)$ -bilinear morphism and $A: \Gamma(VTM^\perp) \times \Gamma(TTM) \rightarrow \Gamma(VTM)$ is a $C(TM)$ -bilinear morphism.

The morphism B is called *(v)-second fundamental form of M and the morphism A_V is the (v)-Weingarten operator*.

Theorem 4.2. *If the linear connection $\tilde{\nabla}$ is metric with respect to g on $VT\tilde{M}$, then the induced linear connections ∇ and $\nabla^\perp$ on VTM and $VTM^\perp$ respectively are metric with respect to g_v and $g_\perp$ respectively.*

Proof. Since $\tilde{\nabla}g=0$, we have $\tilde{\nabla}_X g(Y, Z)=0$ for any $Y, Z \in \Gamma(VTM^\perp)$ and $X \in \Gamma(TTM)$. Hence

$$(4.5) \quad X(g(Y, Z)) = g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z).$$

According of (4.4) we have $X(g(Y, Z)) = g(-A_V X + \nabla_X^\perp Y, Z) + g(Y, -A_Z X + \nabla_X^\perp Z) = g(\nabla_X^\perp Y, Z) + g(Y, \nabla_X^\perp Z)$. Thus $\nabla^\perp g_\perp = 0$.

Similarly, by using (4.3) and taking account that $\tilde{\nabla}g=0$, we have $\nabla g_v = 0$.

We denote by v and h the projectors of TTM to VTM and HTM respectively and by $P: TTM \rightarrow TTM$ the morphism defined by

$$P \left(X_H^\alpha \frac{\delta}{\delta u^\alpha} + X_V^\alpha \frac{\partial}{\partial v^\alpha} \right) = X_H^\alpha \frac{\partial}{\partial v^\alpha} + X_V^\alpha \frac{\delta}{\delta u^\alpha}.$$

Then the induced linear connection ∇ defines as in Sec. 3 two Otsuki connections D and D' with respect to v and h respectively by

$$(4.6) \quad D_X = \nabla_X V,$$

$$(4.7) \quad D'_X = PD_X P = P \nabla_X P h, \quad \forall X \in \Gamma(TTM).$$

Further, by means of the pseudo-Riemannian metric g on $VT\tilde{M}$, we can define a pseudo-Riemannian metric on the whole TTM by

$$(4.8) \quad \tilde{g}(X, Y) = g(\tilde{P}\tilde{h}X, \tilde{P}\tilde{h}Y) + g(\tilde{v}X, \tilde{v}Y).$$

Theorem 4.3. *If $(HT\tilde{M}, \tilde{V})$ is a metric v -Finsler connection on $TT\tilde{M}$ with respect to g on $VT\tilde{M}$, then the induced general vector connection $(HT\tilde{M}, \tilde{v}, \tilde{D})$ and $(HT\tilde{M}, \tilde{h}, \tilde{D}')$ are metric with respect to the prolongation $\tilde{g}$.*

Proof. According to the theorem on 327 [1], we have

$$\begin{aligned} \tilde{D}_z \tilde{g}(X, Y) &= Z \tilde{g}(\tilde{v}X, \tilde{v}Y) - \tilde{g}(\tilde{D}_z X, \tilde{v}Y) - \tilde{g}(\tilde{v}X, \tilde{D}_z Y) = \\ &= Zg(\tilde{v}X, \tilde{v}Y) - g(\tilde{V}_z \tilde{v}X, \tilde{v}Y) - g(\tilde{v}X, \tilde{V}_z \tilde{v}Y) = (\tilde{V}_z g)(\tilde{v}X, \tilde{v}Y) = 0. \end{aligned}$$

$$\begin{aligned} \text{Similarly } (\tilde{D}'_z \tilde{g})(X, Y) &= Z \tilde{g}(\tilde{h}X, \tilde{h}Y) - g(\tilde{D}'_z X, \tilde{h}Y) - g(\tilde{h}X, \tilde{D}'_z Y) = \\ &= Zg(\tilde{P}\tilde{h}X, \tilde{P}\tilde{h}Y) - g(\tilde{P}\tilde{h}\tilde{D}'_z X, \tilde{P}\tilde{h}Y) - g(\tilde{P}\tilde{h}X, \tilde{P}\tilde{h}\tilde{D}'_z Y). \end{aligned}$$

But $\tilde{P}\tilde{h}\tilde{D}'_z X = \tilde{P}\tilde{h}\tilde{P}\tilde{V}_z \tilde{P}\tilde{h}X = \tilde{v}\tilde{P}\tilde{P}\tilde{V}_z \tilde{P}\tilde{h}X = \tilde{V}_z \tilde{P}\tilde{h}X$ and then $(\tilde{D}'_z \tilde{g})(X, Y) = (\tilde{V}_z g)(\tilde{P}\tilde{h}X, \tilde{P}\tilde{h}Y) = 0$.

Now, we are interested in the interrelations of the prolongations of the pseudo-Riemannian metric g_v on TTM and the induced metric on TTM by the prolongation $\tilde{g}$. We denote by $J, J^\perp$ the canonical projections of $VT\tilde{M}$ to VTM and $VTM^\perp$ respectively. Then we have

$$\text{Lemma 4.1. } J\tilde{v}|_{TTM} = v, \quad J^\perp \tilde{v}|_{TTM} = \tilde{v}h, \quad J\tilde{v}h|_{TTM} = 0,$$

$$J\tilde{P}\tilde{h}|_{TTM} = Ph, \quad J^\perp \tilde{P}\tilde{h}|_{TTM} = 0.$$

Proof. For any $X \in \Gamma(TTM)$ we have $J\tilde{v}(X) = J\tilde{v}(hX + vX) = JvX = vX$, since $\tilde{v}v = v$ and $\tilde{v}hX \in VTM^\perp$; $J^\perp \tilde{v}(X) = \tilde{v}h(X)$. Similarly $J\tilde{P}\tilde{h}(hX + vX) = J\tilde{P}\tilde{h}hX = JPhX = PhX$ and then $J^\perp \tilde{P}\tilde{h}|_{TTM} = 0$.

The prolongation of the induced pseudo-Riemannian metric g_v analogous of (4.8) is given by

$$(4.9) \quad \tilde{g}_v(X, Y) = g_v(PhX, PhY) + g_v(vX, vY), \quad \forall X, Y \in \Gamma(TTM).$$

Then we have

Theorem 4.4. *Let $\tilde{M}$ be a real m -dimensional differentiable manifold whose vertical bundle $VT\tilde{M}$ has a pseudo-Riemannian metric g . If $\tilde{g}$ and $\tilde{g}_v$ are the prolongations of g and g_v on $TT\tilde{M}$ and TTM respectively then we have*

$$(4.10) \quad \tilde{g}(X, Y) = \tilde{g}_v(X, Y) + g_\perp(\tilde{v}hX, \tilde{v}hY), \quad \forall X, Y \in \Gamma(TTM).$$

Proof. For any $X, Y \in \Gamma(TTM)$ we have $\tilde{g}(X, Y) = g(\tilde{P}\tilde{h}X, \tilde{P}\tilde{h}Y) + g(\tilde{v}X, \tilde{v}Y) = g(J\tilde{P}\tilde{h}X + J^\perp \tilde{P}\tilde{h}X, J\tilde{P}\tilde{h}Y + J^\perp \tilde{P}\tilde{h}Y) + g(J\tilde{v}X + J^\perp vX, J\tilde{v}Y + J^\perp vY)$.

According to Lemma 4.1 we have $\tilde{g}(X, Y) = g_v(PhX, PhY) + g_v(vX, vY) + g_{\perp}(\tilde{v}hX, \tilde{v}hY) = \tilde{g}_v(X, Y) + g_{\perp}(\tilde{v}hX, \tilde{v}hY)$.

Now we denote by $TTM^{\perp}$ the normal bundle to TM in $T\tilde{M}$ which is the complementary orthogonal subbundle to TTM in $TT\tilde{M}$ with respect to the prolongation $\tilde{g}$.

Remark. If $X \in \Gamma(VTM)$, it doesn't necessarily belong in $TTM^{\perp}$, because from (4.8) we have $\tilde{g}(X, Y) = g(\tilde{P}hX, \tilde{P}hY) + g(\tilde{v}X, \tilde{v}Y)$, $\forall Y \in \Gamma(TTM)$. But $\tilde{P}hX = 0$ and $\tilde{v}X = X$. Hence $\tilde{g}(X, Y) = g(X, \tilde{v}hY) \neq 0$, since $\tilde{v}hY \in VTM^{\perp}$.

In the sequel we use the symbol L and $L^{\perp}$ for the canonical projections of $TT\tilde{M}$ to TTM and $TTM^{\perp}$ respectively. The general connection $(\tilde{D}, \tilde{v})$ which is defined by (3.1) induces a general connection $\tilde{D}$ with respect to endomorphism $L\tilde{v}|_{TTM}$ on TTM and a general connection $\tilde{D}^{\perp}$ with respect to endomorphism $L^{\perp}\tilde{v}|_{TTM^{\perp}}$ on $TTM^{\perp}$ (see [5], [6]). The induced general connections $\tilde{D}, \tilde{D}^{\perp}$ are given by

$$\tilde{D}_X Y = \tilde{D}_X^* Y + H(X, Y): \text{ the Gauss formula } \forall X, Y \in \Gamma(TTM),$$

$$\tilde{D}_X \xi = -G_{\xi}(X) + \tilde{D}_X^{\perp} \xi: \text{ the Weingarten formula } \forall \xi \in \Gamma(TTM^{\perp}),$$

where

$$\tilde{D}_X^* Y = L(\tilde{D}_X Y), \quad H(X, Y) = L^{\perp}(\tilde{D}_X Y), \quad \tilde{D}_X^{\perp} \xi = L^{\perp}(\tilde{D}_X \xi) \quad \text{and} \quad -G_{\xi} X = L(\tilde{D}_X \xi).$$

Similarly, the general connection $(\tilde{D}', \tilde{h})$ defines by (3.2) induces a general connection $\tilde{D}'$ with respect to $L\tilde{h}|_{TTM}$ on TTM and a general connection $\tilde{D}'^{\perp}$ with respect to $L^{\perp}\tilde{h}|_{TTM^{\perp}}$ on $TTM^{\perp}$ by $\tilde{D}'_X Y = \tilde{D}'_X^* Y + H'(X, Y)$, $\tilde{D}'_X \xi = -G'_{\xi}(X) + \tilde{D}'_X^{\perp} \xi$, $\forall X, Y \in \Gamma(TTM)$ and $\xi \in \Gamma(TTM^{\perp})$.

In the sequel, we will look for relations between the induced general connections $\tilde{D}, \tilde{D}'$ and the general connections D, D' of (4.6) and (4.7).

Theorem 4.5. Let $(HT\tilde{M}, \tilde{V})$ be a vector Finsler connection on a manifold $\tilde{M}$ whose vertical bundle has a pseudo-Riemannian structure and M be a submanifold of $\tilde{M}$. Then, if ∇ and $\nabla^{\perp}$ are the induced linear connections on VTM by $\tilde{V}$, the mapping $D^1, D^2: \Gamma(TTM) \times \Gamma(TTM) \rightarrow \Gamma(TTM)$ defined by

$$D_X^1 Y = \nabla_X vY - A_{\tilde{v}hY} X = D_X Y - A_{\tilde{v}hY} X, \quad D_X^2 Y = L(\nabla_X^{\perp} vY + B(X, vY))$$

are general connections on TTM with respect to endomorphism v and $L\tilde{v}$ respectively.

Proof. For any $X, Y, Z \in \Gamma(TTM)$ and $f \in C(TTM)$ we have

$$\begin{aligned} D_{fX+Y}^1 Z &= \nabla_{fX+Y} vZ - A_{\tilde{v}hZ}(fX+Y) = f(\nabla_X vZ - A_{\tilde{v}hZ} X) + \nabla_Y vZ - A_{\tilde{v}hZ} Y = \\ &= fD_X^1 Z + D_Y^1 Z, \end{aligned}$$

$$\begin{aligned} D_X^1(fY+Z) &= \nabla_X v(fY+Z) - A_{\tilde{v}h(fY+Z)} X = \nabla_X f(vY) + \nabla_X vZ - fA_{\tilde{v}hY} X - \\ &- A_{\tilde{v}hZ} X = (Xf)vY + f(D_X^1 Y) + D_X^1 Z. \end{aligned}$$

Similarly $D_{fX+Y}^2 Z = L(\nabla_X^{\perp} \tilde{v}hZ + \nabla_Y^{\perp} \tilde{v}hZ + fB(X, vZ) + B(Y, vZ)) = fD_X^2 Z + D_Y^2 Z$,

$$\begin{aligned} D_X^2(fY+Z) &= L(Xf(\tilde{v}hY) + f\nabla_X^{\perp} \tilde{v}hY + \nabla_X^{\perp} \tilde{v}hZ + fB(X, vY) + B(X, vZ)) = \\ &= (Xf)L\tilde{v}hY + fD_X^2 Y + D_X^2 Z. \end{aligned}$$

The proof is complete.

Theorem 4.6. *Let $(HT\tilde{M}, \tilde{\nabla})$ be a vector Finsler connection on a manifold $\tilde{M}$ whose vertical bundle has a pseudo-Riemannian structure and M be a submanifold of $\tilde{M}$. Then we have*

$$\dot{D} = D^1 + D^2.$$

Proof. Let $X, Y \in \Gamma(TTM)$. Then we have $\dot{D}_X Y = L(\tilde{D}_X Y) = L(\tilde{\nabla}_X \tilde{v} Y) = L(\tilde{\nabla}_X (\tilde{v}(hY + vY))) = L(\tilde{\nabla}_X \tilde{v} hY + \tilde{\nabla}_X vY)$.

Since $vY \in VTM$ and $\tilde{v} hY \in VTM^\perp$ we get $\dot{D}_X Y = L(\nabla_X \tilde{v} hY - A_{\tilde{v} hY} X + \nabla_X vY + B(X, vY)) = \nabla_X vY - A_{\tilde{v} hY} X + L(\nabla_X \tilde{v} hY + B(X, vY)) = (D^1 + D^2)_{XY}$.

According to the Definition 1.3 in [6] the endomorphism of $D^1 + D^2$ is the $v + L\tilde{v}h$. The endomorphism of $\dot{D}$ is $L\tilde{v}|_{TTM}$ but $\forall X \in \Gamma(TTM)$ we have

$$L\tilde{v}(X) = L\tilde{v}(hX + vX) = L\tilde{v}hX + L\tilde{v}vX = L\tilde{v}hX + vX = (L\tilde{v}h + v)X.$$

It is easy to see that the induced general connection D^* on TTM is the induced connection D defined by (4.6), if and only if

$$(4.11) \quad L\nabla_X v hY = A_{\tilde{v} hY} X - B(X, vY).$$

That is $JL\nabla_X \tilde{v} hY = A_{\tilde{v} hY} X$ and $JL\nabla_X \tilde{v} hY = -B(X, vY)$.

Of course, the triplet (HTM, v, D^1) and $(HTM, L\tilde{v}h, D^2)$ are general vector Finsler connections on TTM . If $\dot{T}(X, Y)$ is the torsion tensor of the induced g -connection D^* and $T^1(X, Y)$, $T^2(X, Y)$ are the torsion tensors of the general connections D^1 , D^2 respectively we have $\dot{T}(X, Y) = T^1(X, Y) + T^2(X, Y) = T(X, Y) + T^2(X, Y) + A_{\tilde{v} hY} X - A_{\tilde{v} hY} X$, where $T(X, Y)$ is the torsion tensor of the general connection D .

Theorem 4.7. *Let $(HT\tilde{M}, \tilde{\nabla})$ be a vector Finsler connection on a manifold $\tilde{M}$ whose vertical bundle has a pseudo-Riemannian structure and M be a submanifold of $\tilde{M}$. Then we have*

$$\dot{D}'_X Y = L\tilde{P}(PD'_X Y + B(X, PhY)).$$

Proof. For any $X, Y \in \Gamma(TTM)$ we have $\dot{D}'_X Y = L(\tilde{D}'_X Y) = L(\tilde{P}\tilde{\nabla}_X \tilde{P}hY) = L\tilde{P}(\tilde{\nabla}_X \tilde{P}h(hY + vY))$. Since $\tilde{P}hv = 0$ and $\tilde{P}h h = Ph$ we have $\dot{D}'_X Y = L\tilde{P}\tilde{\nabla}_X PhY$. Using the Gauss formula for the vector Finsler connection $\tilde{\nabla}$ we have $\tilde{D}'_X Y = L\tilde{P}(PD'_X Y + B(X, PhY))$. The endomorphism of $\dot{D}'$ is $L\tilde{h}|_{TTM} = L\tilde{h}h$.

If $\dot{T}'$, T' are the torsion tensors of $\dot{D}'$ and D' respectively we have

$$(4.12) \quad \dot{T}'(X, Y) = L\tilde{P}(PT'(X, Y) + B(X, PhY) - B(Y, PhX)).$$

Taking account of (2.3) we have

$$\begin{aligned} \dot{T}'(X, Y) &= \dot{D}'_X Y - \dot{D}'_Y X - L\tilde{h}h([X, Y]) = L\tilde{P}[PD'_X Y + \\ &+ B(X, PhY) - PD'_Y X - B(Y, PhX)] - L\tilde{h}h[X, Y] = L\tilde{P}[P(T'(X, Y) + \\ &+ h[X, Y]) + B(X, PhY) - B(Y, PhX)] - L\tilde{h}h[X, Y] = L\tilde{P}[PT'(X, Y) + \\ &+ B(X, PhY) - B(Y, PhX)] + L\tilde{P}Ph[X, Y] - L\tilde{h}h[X, Y]. \end{aligned}$$

But $L\tilde{P}Ph = L\tilde{P}\tilde{P}hh = L\tilde{h}h$ and then we have the formula (4.11).

From the above study we see that there is immediate relation between the induced general connections by a vector Finsler connection on a submanifold of a manifold whose vertical bundle has a pseudo-Riemannian structure and the induced general connections by the induced vector Finsler connections ∇ and $\nabla^\perp$.

Received December 15, 1989

University of Thessaloniki,
Department of Mathematics,
Thessaloniki, Greece

REFERENCES

1. Abe N., *General Connections on Vector Bundles*. Kodai Math. J., **8**, 322–329 (1985).
2. Bejancu A., *Geometry of Finsler Subspaces (I)*. An. Șt. Univ. „Al. I. Cuza” Iași, **XXXII**, 2, s. Ia, 69–83 (1986).
3. Bejancu A., *Geometry of Vertical Vector Bundle and its Applications*. Math. Rep. Toyama Univ., **10**, 133–168 (1987).
4. Bejancu A., Otsuki I., *General Finsler Connections on a Finsler Vector Bundle*. Kodai Math. J., **10**, 143–152 (1987).
5. Houh C. S., *Submanifolds in a Riemannian Manifold with General Connections*. Math. J. Okayama Univ., **12**, 1–37 (1963).
6. Nemoto H., *On Differential Geometry of General Connections*. TRU Mathematics, **21**, 67–95 (1985).
7. Okubo T., *Differential Geometry*. Marcel Dekker, New York, 1987.
8. Otsuki T., *On General Connections (I)*. Math. J. Okayama Univ., **9**, 99–164 (1960).
9. Otsuki T., *On General Connections (II)*. Math. J. Okayama Univ., **10**, 113–124 (1961).
10. Yano K., *Structures on Manifolds*. World Scientific Publ. Co., Ltd., Singapore, 1984.

ASUPRA GEOMETRIEI PE FIBRATUL VECTORIAL VERTICAL

(Rezumat)

Se cercetează interrelațiile dintre conexiunea vectorială indusă și conexiunile Finsler vectoriale generale induse pe subvarietăți.

METRICAL KAWAGUCHI CONNECTION OF A LINEAR CONNECTION

BY

CĂTĂLINA NIȚESCU

In paper [1], A. Kawaguchi determines, for a Riemannian space (M, g) , a metrical connection $\overset{k}{\Gamma}$ from a linear arbitrary connection Γ . This connection $\overset{k}{\Gamma}$ is given by the formula (3) and it is called the *metrical Kawaguchi connection* of the connection Γ . For the Finsler spaces, the connection $\overset{k}{\Gamma}$ is considered in the paper [2].

In the present Note we solve the following problem: to establish the set of all linear connections Γ with the property that their Kawaguchi metrical connection $\overset{k}{\Gamma}$ coincides with a metrical connection which has a given torsion tensor field. Some properties of these connections and the semi-symmetric case are studied.

Let (M, g) be a Riemannian space. The following results are known:

1. There exists a unique metrical connection on M with the given torsion tensor field $\overset{\circ}{T}_{jk} = -\overset{\circ}{T}_{kj}$. This is

$$(1) \quad \overset{\circ}{\Gamma}_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \frac{1}{2} g^{il} (g_{lh} \overset{\circ}{T}_{jk}^h - g_{jh} \overset{\circ}{T}_{lk}^h + g_{kh} \overset{\circ}{T}_{jl}^h).$$

2. There exists a unique metrical connection on M with a given torsion vector field $\overset{\circ}{T}_j$. This is

$$(2) \quad \overset{\circ}{\Gamma}_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \frac{1}{n-1} (\overset{\circ}{T}_j \delta_k^i - g_{jk} g^{ih} \overset{\circ}{T}_h), \text{ where } \overset{\circ}{T}_j = \overset{\circ}{T}_{ji}.$$

3. The metrical Kawaguchi connection $\overset{k}{\Gamma}$, of the connection Γ , is given by

$$(3) \quad \overset{k}{\Gamma}_{jh}^i = \Gamma_{jh}^i + \frac{1}{2} g^{im} g_{jm/h},$$

where $"/$ is covariant derivative with respect to Γ .

Further we frequently use the Obata's operators [3]

$$(4) \quad \Omega_{jk}^{ih} = \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}); \quad \Omega^{*ih}_{jk} = \frac{1}{2} (\delta_j^i \delta_k^h + g_{jk} g^{ih}).$$

A first result concerning the metrical Kawaguchi connection is given by

Theorem 1. *The set $\mathcal{K}$ of all linear connections Γ , for which the metrical Kawaguchi connections $\overset{k}{\Gamma}$ coincide with a metrical connection $\overset{\circ}{\Gamma}$ of a given torsion tensor field $\overset{\circ}{T}$ (from the formula (1)) is given by*

$$(5) \quad \Gamma_{jk}^i = \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} + \frac{1}{2} g^{il} (g_{lh} \overset{\circ}{T}_{jk}^h - g_{jh} \overset{\circ}{T}_{lk}^h + g_{kh} \overset{\circ}{T}_{jl}^h) + \Omega^{*im}_{hj} X_{mk}^h,$$

where X_{jk}^i is an arbitrary tensor field.

Proof. We put Γ_{jk}^i in the form

$$(6) \quad \Gamma_{jk}^i = \overset{\circ}{\Gamma}_{jk}^i + A_{jk}^i,$$

where $\overset{\circ}{\Gamma}_{jk}^i$ is written in (1). The tensor field A_{jk}^i is determined by the condition $\overset{k}{\Gamma} = \overset{\circ}{\Gamma}$. Then we get

$$\overset{k}{\Gamma}_{jk}^i = \overset{\circ}{\Gamma}_{jk}^i + A_{jk}^i + \frac{1}{2} g^{im} g_{jm/k}.$$

But

$$g_{jm/k} = g_{jm}^{\circ}{}_{/k} - g_{jh} A_{mk}^h - g_{hm} A_{jk}^h = -g_{jh} A_{mk}^h - g_{hm} A_{jk}^h,$$

since $\overset{\circ}{\Gamma}$ is a metrical connection. Thus we obtain

$$\begin{aligned} g^{im} g_{jm/k} &= -g^{im} g_{jh} A_{mk}^h - g^{im} g_{hm} A_{jk}^h = -\delta_h^i A_{jk}^h - g^{im} g_{jh} A_{mk}^h = \\ &= -\delta_h^i \delta_j^m A_{mk}^h - g^{im} g_{jh} A_{mk}^h = -2\Omega^{*im}_{hj} A_{mk}^h, \end{aligned}$$

$$\overset{k}{\Gamma}_{jk}^i = \overset{\circ}{\Gamma}_{jk}^i + A_{jk}^i - \Omega^{*im}_{hj} A_{mk}^h = \overset{\circ}{\Gamma}_{jk}^i + \delta_h^i \delta_j^m A_{mh}^h -$$

$$- \frac{1}{2} (\delta_h^i \delta_j^m + g^{im} g_{hj}) A_{mk}^h = \overset{\circ}{\Gamma}_{jk}^i + \frac{1}{2} (\delta_h^i \delta_j^m - g^{im} g_{hj}) A_{mk}^h.$$

Therefore

$$\overset{k}{\Gamma}_{jk}^i = \overset{\circ}{\Gamma}_{jk}^i + \Omega^{im}_{hj} A_{mk}^h, \text{ and } \overset{k}{\Gamma}_{jk}^i = \overset{\circ}{\Gamma}_{jk}^i \text{ if and only if } \Omega^{im}_{hj} A_{mk}^h = 0.$$

But this tensor equation has the general solution $A_{jk}^i = \Omega^{*im}_{hj} X_{mk}^h$. Substituting this result in (6) we obtain (5).

Corollaries. 1. Putting in (5) $X_{jk}^i = 0$, we obtain $\Gamma = \overset{\circ}{\Gamma}$. Therefore $\overset{\circ}{\Gamma} \in \mathcal{K}$.

2. If $\Gamma, \bar{\Gamma} \in \mathcal{K}$, then

$$(7) \quad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \Omega^{*im}_{hj} X_{mk}^h.$$

The mapping $\Gamma \rightarrow \bar{\Gamma}$ given by (7) is denoted by $t(X_{jk}^i)$ and it is called a transformation of linear connections of $\mathcal{K}$. Let G be the additive group of tensor fields X_{jk}^i on the manifold M , and $\mathcal{T}$ the group of transformations (7).

The map

$$(8) \quad \varphi : X_{jk}^i \in G \rightsquigarrow \{l(X_{jk}^i) : \Gamma \rightarrow \bar{\Gamma}\} \in \mathcal{C},$$

is an epimorphism. Then $\ker \varphi$ is formed by the kernel of the Obata's operator Ω_{jk}^{*ih} .

3. The tensor field $J_{ijk} = g_{ih} \hat{T}_{jk}^h + g_{jh} \hat{T}_{ki}^h + g_{kh} \hat{T}_{ij}^h$, is an invariant of the group $\mathcal{C}$.

Indeed, from (5) we have for every $X_{jk}^i : J_{ijk} = \hat{J}_{ijk}$.

4. If the set $\mathcal{X}$ contains a symmetric linear connection, then the invariant J_{ijk} vanishes.

5. The set $\mathcal{X}$ contains only one metrical connection.

This is the connection $\bar{\Gamma}$ given by (1).

Proof. If Γ from (5) has the property $g_{ij|k} = 0$, then $g_{hj} X_{ik}^h + g_{hi} X_{jk}^h = 0$ or $\Omega_{hj}^{*im} X_{mk}^h = 0$. This equation has the general solution $X_{jk}^i = \Omega_{hj}^{*im} Y_{mk}^h$, where Y_{jk}^i is an arbitrary tensor field. Substituting in (5) we have $\Gamma = \bar{\Gamma}$.

We remark that if $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} \in \mathcal{X}$ then $\hat{T}_{jk}^i = 0$, $J_{ijk} = 0$.

6. If $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} \in \mathcal{X}$ then there exists in $\mathcal{X}$ another symmetric linear connection. Indeed, in this case, the set $\mathcal{C}$ is given by

$$(9) \quad \Gamma_{jk}^i = \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} + \Omega_{hj}^{*im} X_{mk}^h,$$

where X_{jk}^i is an arbitrary tensor field.

Proof. Taking X_{jk}^i as follows

$$(10) \quad X_{jk}^i = \delta_j^i \sigma_k + 2\delta_k^i \tau_j,$$

we obtain

$$\Gamma_{jk}^i = \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} + \delta_j^i \sigma_k + \delta_k^i \tau_j + g_{jk} g^{im} \tau_m.$$

For $\sigma_j = \tau_j$ the property is evident.

From (5') and (10) we have

$$g_{ij|k} = -2(g_{ij} \sigma_k + g_{jk} \tau_i + g_{ki} \tau_j).$$

If $\tau_j = 0$ one gets a conformal connection and for $\sigma_j = \tau_j$ we have some semi-symmetric linear connections.

Theorem 2. The set $\mathcal{X}'$ of all linear connections Γ for which the metrical Kawaguchi connections $\bar{\Gamma}_k$ coincides with a given metrical semi-symmetric connection $\bar{\Gamma}$ (from (2)) is given by the formula

$$(11) \quad \Gamma_{jk}^i = \begin{Bmatrix} i \\ jk \end{Bmatrix} + \frac{2}{n-1} \Omega_{kj}^{im} \overset{\circ}{T}_m + \Omega_{hj}^{*im} X_{mk}^h,$$

where X_{jk}^i is an arbitrary tensor field.

Remarks. 1. If $\Gamma, \bar{\Gamma} \in \mathcal{K}'$ then we have a mapping $\Gamma \rightarrow \bar{\Gamma}$ given by the formula (7). Let $\mathcal{C}'$ be the set of these transformations. The set $\mathcal{C}'$ and this mapping product is an abelian group isomorphic to the group $\mathcal{C}$.

2. There exists only one metrical semi-symmetric connection in the set $\mathcal{K}'$. This is the connection $\overset{\circ}{\Gamma}$ given by (2).

Received May 10, 1991

Polytechnic Institute, Iassy,
Department of Mathematics

REFERENCES

1. Kawaguchi A., *Beziehung zwischen einer metrischen linearen Übertragung und einer nicht metrischen in einem allgemeinen metrischen Raume*. Proc. Amsterdam Acad., **40**, 3–8 (1937).
2. Miron R., *On a Class of Finsler Connections*. Bul. Inst. Polit. Iași, **XXV(XXIX)**, 3–4, s. I, 53–55 (1979).
3. Nițescu C., *On Conformal Curvature Tensor Relative to the Metric Semi-symmetric Connection*. Bul. Inst. Polit. Iași, **XXV(XXIX)**, 1–2, s. I, 61–63 (1979).

CONEXIUNEA KAWAGUCHI METRICĂ A UNEI CONEXIUNI LINIARE

(Rezumat)

Se consideră un spațiu riemannian (M, g) și se determină mulțimea $\mathcal{K}$ (respectiv $\mathcal{K}'$) a conexiunilor liniare Γ pentru care conexiunea Kawaguchi metrică $\overset{k}{\Gamma}$ coincide cu conexiunea metrică $\overset{\circ}{\Gamma}$ cu tensorul de torsione $\overset{\circ}{T}$ dat (respectiv cu conexiunea metrică semi-simetrică $\overset{\circ}{\Gamma}$) și se găsesc unele proprietăți ale acestor mulțimi.

SEMI-INVARIANT SUBMANIFOLDS OF A NORMAL ALMOST CONTACT METRIC MANIFOLD

BY

C. CĂLIN

0. Introduction. The notion of semi-invariant submanifold (contact CR-submanifold [6]) of a Sasakian manifold has been introduced by A. Bejancu and N. Papaghiuc [4] (K. Yano—M. Kon [6]). The semi-invariant submanifolds of an almost contact metric manifold have been studied in [3].

It is the purpose of this paper to deduce new conditions for the integrability of distributions D , $D \oplus \{\xi\}$ and $D^\perp$ by means of the Nijenhuis tensor N_η .

1. Preliminaries. Let $\tilde{M}$ be a real $2n+1$ -dimensional differentiable manifold and f, ξ, η a tensor field of type (1,1), a vector field and a 1-form, respectively, on $\tilde{M}$, satisfying

$$(1.1) \quad f^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad f(\xi) = 0; \quad \eta \circ f = 0,$$

where I is the identity on the tangent bundle $T\tilde{M}$ of $\tilde{M}$.

Then following D. E. Blair [5], we say that $\tilde{M}$ is an *almost contact manifold* and (f, ξ, η) is the *almost contact structure* on $\tilde{M}$.

Throughout the paper, all manifolds and maps are differentiable of class C^∞ . We denote by $\mathcal{F}(\tilde{M})$ the algebra of the differentiable function on $\tilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\tilde{M})$ -module of sections of vector bundle E over $\tilde{M}$.

Now suppose there exists a Riemannian metric g on $\tilde{M}$ which satisfies

$$(1.2) \quad g(X, Y) = g(fX, fY) + \eta(X)\eta(Y); \quad \eta(X) = g(X, \xi), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

In this case, we say that (f, ξ, η, g) is an *almost contact metric structure* and $\tilde{M}$ is an *almost contact metric manifold*.

Let $\tilde{M}$ be an almost contact metric manifold and M be a m -dimensional submanifold of $\tilde{M}$, so that ξ is tangent to M . We say that M is a semi-invariant submanifold of $\tilde{M}$ if there exist two maximal distributions D and $D^\perp$ so that

$$TM = D \oplus D^\perp \oplus \{\xi\}, \quad fD = D; \quad fD^\perp \subseteq \Gamma(TM^\perp),$$

where (ξ) is the 1-dimensional distribution spanned by ξ on M .

The distributions D and $D^\perp$ are therefore invariant and anti-invariant respectively with respect to f . It is easy to check that the distributions D , $D^\perp$ and (ξ) are mutually orthogonal to each other.

Any vector field X on M is expressed as follows

$$(1.3) \quad X = PX + QX + \eta(X)\xi, \quad \forall X \in \Gamma(TM),$$

where P and Q are the projection morphisms of TM on D and $D^\perp$, respectively. Then we put

$$(1.4) \quad fX = tX + \omega X, \quad \forall X \in \Gamma(TM),$$

where tX belong to $\Gamma(TM)$ and ωX belong to $\Gamma(TM^\perp)$. It follows

$$tX = fPX; \quad \omega X = fQX, \quad \forall X \in \Gamma(TM).$$

We recall the Nijenhuis tensor field N_f defined by

$$(1.5) \quad N_f(X, Y) = [fX, fY] + f^2[X, Y] - f[fX, Y] - f[X, fY], \quad \forall X, Y \in \Gamma(\tilde{M}).$$

The almost contact metric manifold $\tilde{M}$ is called a *normal almost contact metric manifold* if

$$(1.6) \quad N_f(X, Y) + 2d\eta(X, Y)\xi = 0, \quad \forall X, Y \in \Gamma(\tilde{M}).$$

By using (1.3) and (1.4) we infer (see [4])

$$(1.7) \quad t^2X = -PX, \quad \forall X \in \Gamma(D).$$

The Nijenhuis tensor field of t is given by

$$(1.8) \quad N_t(X, Y) = [tX, tY] + t^2[X, Y] - t[tX, Y] - t[X, tY], \quad \forall X, Y \in \Gamma(TM).$$

By using (1.3), (1.5) and (1.8) we obtain

$$(1.9) \quad N_f(X, Y) = N_t(X, Y) - Q([X, Y]) - \omega([tX, Y] + [X, tY]), \\ \forall X, Y \in \Gamma(D \oplus \{\xi\}).$$

Thus from (1.9) we deduce

$$(1.10) \quad N_f(X, Y)^T = N_t(X, Y) - Q([X, Y]), \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}),$$

and

$$(1.11) \quad N_f(X, Y)^\perp = -\omega([tX, Y] + [X, tY]), \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}),$$

where X^T (resp. $X^\perp$) is the tangent part (resp. normal part) of X .

Now, let $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D^\perp)$. By straightforward calculations we infer

$$(1.12) \quad d\eta(X, Y) = \frac{1}{2}\eta([X, Y]).$$

2. Integrability of Distribution on a Semi-Invariant Submanifold of an Almost Contact Metric Manifold. The purpose of this paragraph is to establish necessary and sufficient conditions for integrability of the distributions

D , $D^\perp$, $D \oplus \{\xi\}$. Recall that some related results about the integrability of distributions on a semi-invariant submanifold of an almost contact metric manifold have been established in [3].

Proposition 2.1. *Let M be a semi-invariant submanifold of an almost contact metric manifold $\tilde{M}$. The distribution D is integrable if and only if*

$$(2.1) \quad N_f(X, Y)^T = N_t(X, Y), \quad \forall X, Y \in \Gamma(D),$$

and

$$(2.2) \quad d\eta(X, Y) = 0, \quad \forall X, Y \in \Gamma(D).$$

Proof. It follows from (1.10) and (1.12).

Proposition 2.2. *Let M be a semi-invariant submanifold of an almost contact metric manifold $\tilde{M}$. The distribution D is integrable if and only if*

$$(2.3) \quad N_f(X, Y)^\perp = 0, \quad \forall X, Y \in \Gamma(D),$$

$$(2.4) \quad QN_t(X, Y) = 0, \quad \forall X, Y \in \Gamma(D),$$

$$(2.5) \quad d\eta(X, Y) = 0, \quad \forall X, Y \in \Gamma(D).$$

Proof. Suppose that D is integrable. By using (1.11) and (1.12) we obtain (2.3) and (2.5). Now on taking into account that $D = \text{Im } t$ (see [3]), by using (1.3) and (1.10) we deduce (2.4). Conversely, suppose that (2.3), (2.4) and (2.5) holds. By using (1.11) we infer

$$\omega([tX, Y] + [X, tY]) = 0, \quad \forall X, Y \in \Gamma(D).$$

Thus we obtain

$$Q([fX, Y] + [X, fY]) = 0, \quad \forall X, Y \in \Gamma(D),$$

and consequently

$$Q([fX, fY] + f^2[X, Y]) = 0, \quad \forall X, Y \in \Gamma(D).$$

Hence

$$(2.6) \quad Q(N_f(X, Y)^T) = 0, \quad \forall X, Y \in \Gamma(D).$$

Finally, by using (1.10), (2.4) and (2.6) we obtain

$$(2.7) \quad Q([X, Y]) = 0, \quad \forall X, Y \in \Gamma(D).$$

Thus our assertion follows from (2.5) and (2.7).

By using (1.6) and (1.9) we infer

$$(2.8) \quad N_t(X, Y) - Q([X, Y]) - \omega([tX, Y] + [X, tY]) + 2d\eta(X, Y)\xi = 0, \\ \forall X, Y \in \Gamma(D).$$

Thus we have

Proposition 2.3. *Let M be a semi-invariant submanifold of a normal almost contact metric manifold $\tilde{M}$. The distribution D is integrable if and only if*

$$(2.9) \quad N_t(X, Y) = 0, \quad \forall X, Y \in \Gamma(D).$$

Similarly we state

Proposition 2.4. *Let M be a semi-invariant submanifold of a normal almost contact metric manifold $\tilde{M}$. The distribution $D \oplus (\xi)$ is integrable if and only if*

$$N_t(X, Y) + 2d\eta(X, Y)\xi = 0, \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}).$$

Now, we take $X, Y \in \Gamma(D^\perp \oplus \{\xi\})$. By using (1.7) and (1.8) we infer

$$(2.10) \quad N_t(X, Y) = -P([X, Y]), \quad \forall X, Y \in \Gamma(D^\perp \oplus \{\xi\}).$$

By using (1.12) and (2.10) we obtain

Proposition 2.5. *Let M be a semi-invariant submanifold of an almost contact metric manifold $\tilde{M}$. The distribution $D^\perp$ is integrable if and only if*

$$N_t(X, Y) = 0, \quad \forall X, Y \in \Gamma(D^\perp),$$

and

$$d\eta(X, Y) = 0, \quad \forall X, Y \in \Gamma(D \oplus \{\xi\}).$$

Received September 8, 1991

Polytechnic Institute, Jassy
Department of Mathematics

REFERENCES

1. Bejancu A., *CR-submanifold of a Kähler Manifold* (I). Proc. Amer. Soc., **69**, 135–142 (1978).
2. Bejancu A., *Geometry of CR-submanifolds*. D. Reidel Publ. Comp., Dordrecht, 1986.
3. Bejancu A., *Semi-invariant Submanifolds of an Almost Contact Metric Manifold*. An. Șt. Univ. Iași, Supl. **XXVII**, 1, I-a, 17–21 (1981).
4. Bejancu A., Papaghiuc N., *Semi-invariant Submanifolds of a Sasakian Manifold*. An. Șt. Univ. „Al. I. Cuza” Iași, Supl. **XVII**, 1, I-a, 163–170 (1981).
5. Blair D. E., *Contact Manifolds in Riemannian Geometry*. Lecture Notes in Math., 509, Springer-Verlag, Berlin, 1976.
6. Yano K., Kon M., *Contact CR-submanifolds*. Kodai Math. J., 238–252 (1982).
7. Yano K., Kon M., *Contact CR-submanifolds of Kählerian and Sasakian Manifold*. Birkhäuser, Boston, 1983.

CR-SUBVARIETĂȚI ALE VARIETĂȚILOR APROAPE DE CONTACT METRICE NORMALE

(Rezumat)

Se studiază integrabilitatea distribuțiilor unei CR-subvarietăți M a varietății aproape de contact $\tilde{M}$. Condițiile au fost stabilite cu ajutorul tensorului Nijenhuis atașat tensorului f .

QUELQUES CONSIDÉRATIONS SUR LES ESPACES T -MÉTRIQUES COMPACTS

PAR

TEMISTOCLE BÎRSAN

1. Introduction. La notion d'espace T -métrique a été introduite dans [1].

Définition. On dit que $d: X \times X \rightarrow \mathbb{R}_+$ est une métrique sur X par rapport à la fonction $T: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ si

I. $d(x, y) = 0 \Leftrightarrow x = y$,

II. $d(x, y) = d(y, x)$, pour tous $x, y \in X$,

III. $d(x, z) \leq T(d(x, y), d(y, z))$, pour tous $x, y, z \in X$, $x \neq y \neq z \neq x$. On dit que d est une T -métrique sur X et que T est une t -fonction (c-à-d. fonction triangulaire). Le triplet (X, d, T) est appelé *espace T -métrique*.

Si $T(a, b) = a + b$, alors (X, d, T) est un espace métrique classique. Dans le cas où $T(a, b) = \max(a, b)$, (X, d, T) est un espace métrique non-archimédien. La structure de l'espace T -métrique dépend de la t -fonction T . Les propriétés suivantes de T seront importantes dans ce qui suit :

1° $\lim_{a \rightarrow 0, b \rightarrow 0} T(a, b) = 0$,

2° $T(0, b) \leq b$, pour tout $b \in \mathbb{R}_+$,

3° $t \rightarrow T(t, b)$ est une application scs (c-à-d. semi-continue supérieure) au point $t = 0$, quel que soit $b \in \mathbb{R}_+$.

On désigne par $S(x, r)$ et U_ε les ensembles $\{y; d(x, y) < r\}$ et $\{(x, y); d(x, y) < \varepsilon\}$, respectivement. Une topologie est compatible avec la T -métrique d si l'ensemble $\{S(x, r); r > 0\}$ est un système fondamental de voisinages de x , quel que soit $x \in X$. S'il existe, une telle topologie est unique. L'espace (X, d, T) est métrisable s'il existe une métrique (classique) sur X qui a $\{U_\varepsilon; \varepsilon > 0\}$ pour système fondamental d'entourages.

Théorème A ([2], Théorème 1). *Si (X, d, T) est un espace T -métrique dont la t -fonction T satisfait à la condition 1°, alors cet espace est métrisable.*

Théorème B ([2], Théorème 2). *Soit (X, d, T) un espace T -métrique dont la t -fonction T satisfait aux conditions 2° et 3°. Alors*

1) *il existe une topologie sur X compatible avec d , et*

2) *les sphères $S(x, r)$ sont des ensembles ouverts par rapport à elle.*

Remarques 1. La condition 1° n'assure pas que les sphères $S(x, r)$ sont ouvertes ([2], Exemple).

2. Si (X, d, T) admet une topologie compatible avec T , alors celle-ci a des bases locales dénombrables dans tous les points de X .

3. Si T vérifie la condition 1°, alors les suites convergentes de l'espace (X, d, T) sont aussi des suites de Cauchy.

2. Le but de ce travail est de préciser les conditions que nous devons imposer à la t -fonction T pour que les résultats principaux des espaces métriques compacts restent valables et dans le cadre des espaces T -métriques. Normalement, ces conditions seront choisies de sorte que les généralisations envisagées se démontrent comme dans le cas classique. C'est pourquoi nous omettons la plupart des démonstrations. Pourtant, quelques affirmations seront démontrées en détail pour voir le rôle de la t -fonction T et des hypothèses considérées.

Théorème 1. *Si l'espace (X, d, T) admet une topologie compatible avec la T -métrique d , on a l'équivalence des conditions :*

- (i) *l'espace est dénombrablement compact, et*
- (ii) *l'espace est compact par suites.*

Démonstration. En effet, on sait que (i) $\Rightarrow$ (ii) est vraie pour les espaces topologiques ayant une base locale dénombrable dans tous leurs points, tandis que (ii) $\Rightarrow$ (i) est vraie même dans les espaces topologiques quelconques [3].

Dans les hypothèses des Théorèmes A et B, l'espace (X, d, T) admet une topologie compatible avec d et, par conséquent, on a

Corollaire 1. *Soit (X, d, T) un espace T -métrique dont T vérifie la condition 1°. Alors les affirmations (i) et (ii) sont équivalentes.*

Corollaire 2. *Soit (X, d, T) un espace T -métrique dont T vérifie les conditions 2° et 3°. Alors les propriétés (i) et (ii) sont équivalentes.*

Par le même voie comme dans [3], on établit la

Proposition 1. *Soit (X, d, T) un espace T -métrique dont T satisfait à la condition 1° et l'espace est dénombrablement compact. Alors, on a :*

- (i) *(X, d, T) est totalement borné,*
- (ii) *(X, d, T) est séparable.*

Proposition 2. *Soit (X, d, T) un espace T -métrique dont T satisfait aux conditions 1°, 2° et 3°. Alors les propriétés suivantes :*

- (i) *X est un espace de Lindelöf,*
- (ii) *X est séparable, et*
- (iii) *X possède une base dénombrable*

sont équivalentes.

Démonstration. On prouve habituellement que (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). D'après le Théorème B, les hypothèses 2° et 3° assurent que les sphères $S(x, r)$ sont ouvertes. Cette chose intervient lorsqu'on établit que (ii) $\Rightarrow$ (iii) (les autres implications peuvent être démontrées seulement dans l'hypothèse 1°). En effet, $A = \{x_0, x_1, \dots, x_n, \dots\}$ étant une partie dense de X , on considère $\mathcal{B} = \{S(x_n, r) ; n \text{ nombre naturel et } r > 0 \text{ rationnel}\}$. En vertu de 2° et 3°, $\mathcal{B}$ est une famille d'ensembles ouverts. Soient D un ensemble ouvert arbitraire et $x \in D$. Il existe un $r > 0$ et rationnel tel que $S(x, r) \subset D$. D'autre part, grâce à l'hypothèse (1), il y a $\delta > 0$ et réel de sorte que $t < \delta$ entraîne $T(t, t) < r$.

Choisissons un nombre rationnel $s > 0$ et tel que $s < \delta$ et un point $x_n \in S(x, s) \cap A$. Pour n'importe quel $z \in S(x_n, s)$, on a

$$d(x, z) \leq T(d(x, x_n), d(x_n, z)) < r,$$

car $d(x, x_n) < s$, $d(x_n, z) < s$ et $s < \delta$. Donc $z \in S(x, r)$ et, par conséquent, $S(x_n, s) \subset S(x, r) \subset D$. Alors $x \in S(x_n, s) \subset D$ et $S(x_n, s) \in \mathcal{B}$ c'est-à-dire $\mathcal{B}$ est une base dénombrable de l'espace X .

Théorème 2. Soit (X, d, T) un espace T -métrique dont T a les propriétés 1°, 2° et 3°. Les conditions suivantes :

- (i) l'espace T -métrique est compact, et
- (ii) l'espace T -métrique est dénombrablement compact

sont équivalentes.

Démonstration. (i) $\Rightarrow$ (ii) est triviale. Réciproquement, grâce aux Propositions 1 et 2, (ii) entraîne que (X, d, T) est un espace de Lindelöf et, par conséquent, il est compact.

Des Théorèmes 1 et 2 il suit le

Théorème 3. Soit (X, d, T) un espace T -métrique dont T vérifie les conditions 1°, 2° et 3°. Alors, les propriétés

- (i) (X, d, T) est compact, et
- (ii) (X, d, T) est compact par suites

sont équivalentes.

Les résultats qui expriment le rapport entre les espaces métriques compacts et complets se généralisent facilement aux espaces T -métriques (dans des conditions convenables sur la T -métrique).

Proposition 3. Soit (X, d, T) un espace T -métrique de sorte que T vérifie la condition 1°. Si l'espace est compact, alors il est totalement borné et complet.

Démonstration. En vertu de la Proposition 1, l'espace est totalement borné. Parce que la t -fonction T a la propriété 1°, on peut prouver que (X, d, T) est complet d'une manière usuelle.

Proposition 4. Soit (X, d, T) un espace T -métrique dont T satisfait à la condition 1°. Si l'espace est totalement borné et complet, alors est compact par suites.

Le Théorème 3 et les Propositions 3 et 4 ont pour conséquence le

Théorème 4. Soit (X, d, T) un espace T -métrique dont T satisfait aux conditions 1°, 2° et 3°. L'espace est compact si et seulement si il est totalement borné et complet.

Remarque. Les conditions 1°, 2° et 3° ne sont pas restrictives. Elles sont vérifiées par les t -fonctions $T(a, b) = a + b$, $T(a, b) = \max(a, b)$, $T(a, b) = a + b + ab$, etc. Pour la première t -fonction, la T -métrique d devient une métrique classique ordinaire et les résultats précédents coïncident avec ceux classiques correspondants.

Reçue le 24 Mars, 1991

Institut Polytechnique, Jassy,
Chaire de Mathématiques

BIBLIOGRAPHIE

1. Bîrsan T., Une application d'un théorème de J. Aczél. Publ. Math. (Debrecen), **31**, 113—118 (1984).

2. Bîrșan T., *Quelques propriétés des espaces T -métriques*. Bul. Inst. polit. Iași, XXXII(XXXVI), 1-4, s. I, 19-23 (1986).
3. Kuratowski C., *Topologie*. T. I-II, Warszawa, 1958-1961.

CÎTEVA CONSIDERAȚII ASUPRA SPAȚIILOR T -METRICE COMPACTE

(Rezumat)

O T -metrică pe o mulțime X este o funcție $d: X \times X \rightarrow X$ ce satisface condițiile I-III. Dacă funcția $T: \mathbf{R} \times \mathbf{R}_+$ ce intervine are proprietățile 1°-3°, atunci se obțin următoarele generalizări ale unor rezultate cunoscute: 1) un spațiu T -metric (X, d, T) este compact dacă și numai dacă este compact prin șiruri (Teorema 3); 2) (X, d, T) este compact dacă și numai dacă este total mărginit și complet (Teorema 4).

SUR L'ADMISSIBILITÉ D'UNE PAIRE D'ESPACES DE FONCTIONS RÉGLÉES PAR RAPPORT À UN OPÉRATEUR INTÉGRAL NON LINÉAIRE DE TYPE VOLTERRA

PAR

GH. BANTAȘ

Les problèmes d'admissibilité sont importants tant en théorie des opérateurs ([6]), où ils nous donnent des informations sur les propriétés des opérateurs par rapport auxquels les diverses paires d'espaces sont admissibles, qu'en théorie des équations différentielles et intégrales ([4], [2], [1]), où, en même temps que les théorèmes de point fixe, ils servent à la démonstration de l'existence et, éventuellement, de l'unicité des solutions dans divers espaces de fonctions.

Nous rappelons que deux espaces E et F s'appellent admissibles par rapport à un opérateur T si $TE \subseteq F$.

Dans cet article nous allons démontrer un théorème d'admissibilité de deux espaces des fonctions réglées sur le demi-axe $[a, +\infty[$ par rapport à un opérateur intégral non linéaire de type Volterra.

Soit B l'espace des fonctions $x: [a, +\infty[\rightarrow \mathbb{R}^p$ bornées sur $[a, +\infty[$. Dans l'espace B on considère le sous-espace D de Banach des fonctions réglées sur le demi-axe $[a, +\infty[$. La norme de ce sous-espace est définie par la formule

$$\|x\| = \sup_{t \geq a} \|x(t)\|,$$

où $\|x(t)\|$ est la longueur euclidienne du vecteur $x(t)$.

Par définition une application $x: [a, +\infty[\rightarrow \mathbb{R}^p$ s'appelle *régulée* sur $[a, +\infty[$ si elle est réglée sur chaque segment $[a, b] \subset [a, \infty[$ ([5], [3]).

Soient S la boule de centre O et de rayon r de l'espace D , Σ la boule de centre O et de rayon r de l'espace $\mathbb{R}^p$ et $\Delta = \{(t, s) / a \leq s \leq t < \infty\}$.

Considérons l'opérateur intégral non linéaire de type Volterra

$$(Kx)(t) = \int_a^t K(t, s, x(s)) ds,$$

où $K: \Delta \times \Sigma \rightarrow \mathbb{R}^p$ et $x \in D \cap S$.

Le problème se pose de trouver des conditions qui assurent l'admissibilité des espaces $D \cap S$ et D par rapport à l'opérateur K , c'est-à-dire l'inclusion $K(D \cap S) \subset D$.

Une réponse à ce problème est donnée par le théorème suivant.

Théorème. Supposons que le noyau de l'opérateur K satisfait les conditions suivantes :

1) Pour chaque $(t, s) \in \Delta$ et $x, y \in \Sigma$, a lieu l'inégalité

$$\|K(t, s, x) - K(t, s, y)\| \leq A(t, s) \|x - y\|,$$

où $A: \Delta \rightarrow \mathbb{R}$ est une fonction positive, sommable par rapport à $s \in [a, t]$ pour

chaque $t \geq a$ fixé et où $\int_a^t A(t, s) ds \leq r' \in \mathbb{R}_+$ sur $[a, \infty[$.

2) Pour chaque $t \geq a$ fixé, l'application $K(t, \cdot, \cdot)$ est sommable sur $[a, t]$.

3) Pour chaque $b \geq a$ et $x \in \Sigma$ fixé les fonctions $\int_a^b K(\cdot, s, x) ds$ et $\int_a^b K(\cdot, s, x) ds$

appartiennent à D .

Alors l'opérateur K existe et les espaces $D \cap S$ et D sont admissibles par rapport à cet opérateur : $K(D \cap S) \subseteq D$.

Démonstration. Montrons d'abord que l'opérateur K existe. Dans ce but il suffit de démontrer que pour tout $t \geq a$ fixé et pour chaque $x \in D \cap S$ fixé l'application $s \rightarrow K(t, s, x(s))$ est sommable par rapport à $s \in [a, t]$.

Soient $t \geq a$ un nombre fixé et $x \in D \cap S$ une fonction fixée. En tenant compte de la condition 1) on a

$$\begin{aligned} \|K(t, s, x(s))\| &\leq \|K(t, s, x(s)) - K(t, s, 0)\| + \|K(t, s, 0)\| \leq A(t, s) \|x(s)\| + \\ &+ \|K(t, s, 0)\| \leq rA(t, s) + \|K(t, s, 0)\|. \end{aligned}$$

D'ici et des conditions 1) et 2) il s'en suit que pour chaque $t \geq a$ fixé et pour chaque $x \in D \cap S$ fixé l'application $s \rightarrow K(t, s, x(s))$ est sommable sur $[a, t]$. Donc l'opérateur K existe.

Montrons maintenant que pour chaque $x \in D \cap S$, a lieu l'affirmation : $Kx \in B$. Soit $x \in D \cap S$ fixé. De la condition 1) on a

$$\begin{aligned} \|(Kx)(t)\| &\leq \left\| \int_a^t [K(t, s, x(s)) - K(t, s, 0)] ds \right\| + \left\| \int_a^t K(t, s, 0) ds \right\| \leq \\ &\leq \int_a^t \|K(t, s, x(s)) - K(t, s, 0)\| ds + \left\| \int_a^t K(t, s, 0) ds \right\| \leq \int_a^t A(t, s) \|x(s)\| ds + \\ &+ \left\| \int_a^t K(t, s, 0) ds \right\| \leq rr' + \left\| \int_a^t K(t, s, 0) ds \right\|, \quad (t \geq a). \end{aligned}$$

De la condition 3) il résulte, en particulier, que $\int_a^t K(t, s, 0) ds \in B$. Donc, il y a un $M \in \mathbb{R}_+$ tel que

$$\left\| \int_a^t K(t, s, 0) ds \right\| \leq M, \quad (t \geq a).$$

D'ici et de l'inégalité précédente on obtient

$$\|(\mathbf{K}x)(t)\| \leq rr' + M, \quad (t \geq a).$$

Donc $\mathbf{K}x \in B$.

Montrons que pour chaque $x \in D \cap S$ a lieu l'affirmation $\mathbf{K}x \in D$. Dans ce but il faudra démontrer que la restriction de $\mathbf{K}x$ à tout sous-intervalle $[a, b] \subset [a, \infty[$ est une fonction réglée. Ceci sera réalisée en deux étapes. Dans la première étape l'affirmation s'établit pour les fonctions en escalier tandis que dans la seconde pour celles réglées.

Soit $u : [a, b] \rightarrow \mathbb{R}^p$ où $\sup_{[a, b]} \|u(t)\| < r$, une fonction en escalier. Alors il y a une division du segment $[a, b]$,

$$a = b_0 < b_1 < \dots < b_n = b,$$

telle que, $u(t) = c_k \in \mathbb{R}^p$ pour tout $t \in [b_{k-1}, b_k]$. Soit $t \in [a, b]$. Alors il existe un intervalle $[b_{m-1}, b_m]$ tel que $t \in [b_{m-1}, b_m]$. On a

$$(\mathbf{K}u)(t) = \sum_{k=1}^{m-1} \int_{b_{k-1}}^{b_k} K(t, s, c_k) ds + \int_{b_{m-1}}^t K(t, s, c_m) ds.$$

D'ici et de la condition 3) on déduit que l'application $\mathbf{K}u$ est réglée sur $[a, b]$.

Considérons maintenant une fonction réglée $x \in D \cap S$. Pour simplifier, nous notons toujours avec x la restriction de x à $[a, b]$. D'après un théorème de Bourbaki ([3], [5]) pour la restriction x il y a une suite de fonctions en escalier $u_n : [a, b] \rightarrow \mathbb{R}^p$ qui converge uniformément sur $[a, b]$ à la restriction $x : u_n(s) \rightrightarrows x(s)$. D'ici et du fait que $\sup_{t \geq a} \|x(t)\| < r$ il s'en suit que nous pouvons supposer que $\sup_{[a, b]} \|u_n(t)\| < r$ pour chaque $n \geq 1$. En utilisant la condition 1), on a

$$\begin{aligned} \|(\mathbf{K}x)(t) - (\mathbf{K}u_n)(t)\| &= \left\| \int_a^t K(t, s, x(s)) ds - \int_a^t K(t, s, u_n(s)) ds \right\| \leq \\ &\leq \int_a^t \|K(t, s, x(s)) - K(t, s, u_n(s))\| ds \leq \int_a^t A(t, s) \|x(s) - u_n(s)\| ds \leq \\ &\leq \sup_{s \in [a, b]} \|x(s) - u_n(s)\| \int_a^t A(t, s) ds \leq r' \sup_{s \in [a, b]} \|x(s) - u_n(s)\|. \end{aligned}$$

D'ici il résulte que pour la restriction x a lieu l'affirmation suivante :

$$(1) \quad (\mathbf{K}u_n)(t) \rightrightarrows (\mathbf{K}x)(t) \quad \text{sur } [a, b].$$

D'après ce qui a été démontré dans l'étape précédente il s'en suit que les fonctions Ku_n sont des fonctions réglées sur $[a, b]$. D'ici, de l'affirmation (1) et de la propriété de convergence uniforme des suites de fonctions réglées ([3], [5]) il résulte que la fonction Kx est réglée sur $[a, b]$. Comme l'intervalle $[a, b]$ a été choisie d'une manière arbitraire il s'en suit que la fonction Kx , où $x \in D \cap S$ est réglée sur chaque $[a, b] \subset [a, \infty[$. Par conséquent, $Kx \in D$ pour chaque $x \in D \cap S$. Donc $K(D \cap S) \subset D$.

Requ le 16 Septembre, 1991

Université „Al. I. Cuza”, Jassy,
Séminaire Mathématique „Al. Myller”

BIBLIOGRAPHIE

1. Bantaș Gh., *Équations intégral-fonctionnelles de Volterra à solutions dans certains espaces de fonctions réglées*. Proc. Inst. Math. Iași, Ed. Acad. Buc., pp. 103—107 (1976).
2. Corduneanu C., *Problèmes globaux dans la théorie des équations intégrales de Volterra*. Ann. Mat. Pura ed Appl., **67**, 4, 349—363 (1965).
3. Dieudonné J., *Éléments d'analyse* (1). Gauthier Villars, Paris, 1969.
4. Massera J. L., Schäffer J. J., *Linear Differential Equations and Function Spaces*. Academic Press, New York—London, 1966.
5. Nicolescu M., *Analiză matematică*. T. 2, Ed. tehn., Buc., 1958.
6. Забрейко П. П., Кошелев А. И., Красносельски М. А., Михлин С. Г., Раковщик Л. С., Стеценко В. Я., *Интегральные уравнения*. Справочная математическая библиотека, Москва, 1968.

DESPRE ADMISIBILITATEA UNEI PERECHI DE SPAȚII DE FUNCȚII RIGLATE ÎN RAPORT CU UN OPERATOR INTEGRAL NELINIAR DE TIP VOLTERRA

(Rezumat)

Se dau condiții suficiente pentru ca un operator integral neliniar de tip Volterra

$$(Kx)(t) = \int_a^t K(t, s, x(s)) ds$$

să realizeze incluziunea $K(D \cap S) \subset D$, unde D este spațiul Banach al funcțiilor riglate și mărginite $x: [a, \infty[\rightarrow \mathbb{R}^p$ dotat cu norma convergenței uniforme iar $S \subset D$ este sfera de centru O și rază $r > 0$.

SUR LES SYSTÈMES D'ÉQUATIONS DIFFÉRENTIELLES À PETIT PARAMÈTRE

PAR

C. SIMIRAD

1. Introduction. Les phénomènes nonlinéaires intervenant dans certains système d'équations différentielles apparaissent quelque fois par des termes nonlinéaires multipliés par un petit paramètre. Ce fait conduit à l'idée que les termes respectifs peuvent-être négligés. Pourtant, une analyse attentive s'impose, afin de montrer que cette omission est justifiée.

Dans ce qui suit nous allons utiliser des nouveaux résultats dans la méthode du petit paramètre, en offrant un moyen efficace dans l'étude pratique. Nous allons considérer un système différentiel générateur ayant des perturbations contenant comme facteur un petit paramètre, dont nous allons associer un système différentiel linéaire. Des hypothèses simples sur le système linéaire assurent l'existence et l'unicité d'une solution bornée pour le système générateur, solution approximant celle du système perturbé.

Soit le système différentiel à petit paramètre

$$(1) \quad \dot{x}(t) = f(t, x(t)) + \varepsilon h(t, x(t), \varepsilon)$$

et le système générateur associé

$$(2) \quad \dot{x}(t) = f(t, x(t)).$$

Comme hypothèse générale on suppose qu'il existe une matrice $g: \mathbb{R} \rightarrow \mathbb{R}$, $g(t) = (g_{ij}(t))_{n \times n}$ des fonctions continues satisfaisant les inégalités

$$(I) \quad \begin{aligned} \frac{\partial f_i}{\partial x_i}(t, x) &\leq g_{ii}(t), \quad i=1, \dots, n, \quad \forall(t, x); \\ \left| \frac{\partial f_i}{\partial x_j}(t, x) \right| &\leq g_{ij}(t), \quad i \neq j, \quad \forall(t, x), \end{aligned}$$

Nous allons associer au système générateur (2) le système différentiel linéaire

$$(3) \quad \dot{y}(t) = g(t)y(t) + 1,$$

où le symbole 1 représente la matrice-colonne avec tous les éléments 1.

Supposons que le système différentiel (1) est dans les conditions habituelles de régularité, c'est à dire avec les dérivées partielles des fonctions f et h continues. De plus, ces deux fonctions sont ω -périodiques dans la variable t .

Nous allons utiliser un résultat du [3] affirmant que si on a deux systèmes différentiels $\dot{x}(t)=a(t)x(t)$ et $\dot{y}(t)=b(t)y(t)$ avec les matrices fondamentales des solutions respectivement $A(t, s)$ et $B(t, s)$ alors, pour avoir les inégalités $|A_{ij}(t, s)| \leq B_{ij}(t, s)$ il faut et il suffit d'avoir les inégalités $a_{ii}(t) \leq b_{ii}(t)$ et $|a_{ij}(t)| \leq b_{ij}(t)$, $\forall i \neq j$, $i, j=1, \dots, n$, $t \in \mathbb{R}$, $t \geq s$.

De même, nous allons utiliser un résultat du [2], qui montre que si le système (3) a une solution positive et bornée à l'avenir, alors, le système (2) a une solution bornée unique, qui de plus est ω -périodique.

2. Le principal résultat. Compte tenu de ce que nous avons précisé ci-dessus nous pouvons énoncer le

Théorème 1. *Si les inégalités (I) sont remplies et le système (3) a une solution positive, bornée à l'avenir, alors le système générateur (2) a une solution bornée unique et ω -périodique $x_0(t)$ et il existe $\varepsilon_0 > 0$ tel que pour $|\varepsilon| < \varepsilon_0$ le système (1) a une solution ω -périodique $x(t, \varepsilon)$ ayant la propriété que $\lim_{\varepsilon \rightarrow 0} x(t, \varepsilon) = x_0(t)$.*

Démonstration. Pour mieux comprendre la démonstration, nous allons suivre au début l'essentiel de la méthode du petit paramètre (voir [1], p. 240). Du [2] nous avons assurée l'existence et l'unicité d'une solution bornée et ω -périodique du système (2). Soit cette solution $x_0(t)$ dans la condition initiale $x_0(0)=p_0$. Considérons que la solution du système (1) a la forme $x(t, p, \varepsilon)$ dans la condition initiale $x(0, p, \varepsilon)=p$. Évidemment, la solution x sera ω -périodique si $x(\omega, p, \varepsilon)=x(0, p, \varepsilon)=p$ et de plus $x(t, p, 0)=x_0(t, p)$, où $x_0(t, p)$ est la solution du système générateur ayant $x_0(0, p)=p$.

En posant $u(p, \varepsilon)=x(\omega, p, \varepsilon)-p$ on observe que $u(p_0, 0)=0$ et si

$$(4) \quad \det \frac{\partial u}{\partial p} \Big|_{(p_0, 0)} \neq 0,$$

alors, conformément au théorème des fonctions implicites, $\exists \varepsilon_0 > 0$ tel que pour $|\varepsilon| < \varepsilon_0$ il existe $p(\varepsilon)$ continue, avec $p(0)=0$ et satisfaisant $u(p(\varepsilon), \varepsilon)=0$.

La solution $x(t, p(\varepsilon), \varepsilon)$ du système différentiel (1) sera la solution ω -périodique que nous cherchons. Donc, nous devons assurer la condition (4) et la démonstration sera finie.

Du fait que $\frac{\partial u}{\partial p} = \frac{\partial}{\partial p} x(\omega, p, \varepsilon) - 1$ il résulte que la condition (4) est équivalente au fait que la matrice $\frac{\partial}{\partial p} x(\omega, p, \varepsilon)$ ne doit pas avoir $\lambda = 1$ comme valeur propre.

De (2) on a $\dot{x}_0(t, p)=f(t, x_0(t, p))$ et par la suite nous aurons l'égalité suivante

$$\frac{d}{dt} \frac{\partial x_0(t, p)}{\partial p} = \frac{\partial}{\partial x} f(t, x_0(t, p)) \frac{\partial}{\partial p} x_0(t, p),$$

et pour $p=p_0$ on obtient

$$\frac{d}{dt} \frac{\partial x_0(t, p_0)}{\partial p} = \frac{\partial}{\partial x} f(t, x_0(t, p_0)) \frac{\partial}{\partial p} x_0(t, p_0).$$

En d'autres termes, la matrice $\frac{\partial}{\partial p} x_0(t, p_0)$ est une matrice des solutions du système linéaire

$$(5) \quad \dot{z}(t) = f(t, x_0(t))z(t).$$

Puisque $x_0(0, p) = p$ il résulte que $\frac{\partial}{\partial p} x_0(t, p) \Big|_{t=0} = I$, I étant la matrice unité et par la suite $\frac{\partial}{\partial p} x_0(t, p)$ est justement la matrice fondamentale des solutions du système en variations (5) correspondant à la solution du système générateur.

Du fait que $x_0(t)$ est ω -périodique, il résulte que $\frac{\partial}{\partial x} f(t, x_0(t, p))$ est ω -périodique et de la suite d'égalités

$$\frac{\partial}{\partial p} x(\omega, p, \varepsilon) \Big|_{\substack{\varepsilon=0 \\ p=p_0}} = \frac{\partial}{\partial p} x(\omega, p, 0) \Big|_{p=p_0} = \frac{\partial}{\partial p} x(t, p, 0) \Big|_{\substack{t=\omega \\ p=p_0}} = \frac{\partial}{\partial p} x_0(t, p) \Big|_{\substack{t=\omega \\ p=p_0}}$$

on obtient que la matrice $\frac{\partial}{\partial p} x_0(t, p) \Big|_{\substack{t=\omega \\ p=p_0}}$ est justement la matrice de monodromie du système (5). La matrice de monodromie n'a pas la valeur propre $\lambda=1$ si et seulement si le système en variations (5) n'a pas de solution ω -périodique.

Supposons par absurde que le système en variations (5), dans les hypothèses du Théorème 1, a une solution ω -périodique $z(t)$. Notons par $F(t, s)$ la matrice fondamentale des solutions du système (5) et par $G(t, s)$ la matrice fondamentale du système (3). Puisque on a les inégalités (I) nous aurons $|F(t, s)| \leq G(t, s)$, $\forall t \geq s$. La solution $z(t)$ a la forme $z(t) = F(t, s)z(s)$ et donc il résulte l'inégalité $|z(t)| \leq |F(t, s)| |z(s)|$.

La fonction $z(t)$ étant ω -périodique est bornée et donc il existe $M > 0$ tel que $|z(s)| \leq M$, $\forall s \in \mathbb{R}$. Puisque le système (3) a une solution positive et bornée à l'avenir, il résulte (voir [2]) que $\liminf_{s \rightarrow -\infty} |G(t, s)| = 0$.

De l'inégalité $|z(t)| \leq |F(t, s)| |z(s)| \leq G(t, s)M$ il résulte que $\lim_{t \rightarrow -\infty} z(t) = 0$ ce qui implique le fait que $z(t) = 0$, $\forall t \in \mathbb{R}$. Donc le système en variations, dans les conditions du Théorème 1, n'a pas des solutions ω -périodiques, sauf la solution banale. La démonstration est finie.

À la différence de la méthode du petit paramètre, le Théorème 1 a assuré l'existence et l'unicité d'une solution ω -périodique pour le système générateur. De même, nous avons renoncé à l'hypothèse que le système en variations n'a pas des solutions ω -périodiques, hypothèse très difficile à vérifier.

Dans les cases pratiques, on cherche, s'il est possible, de trouver une matrice g constante, telle que les inégalités (I) soient remplies. Dans ce qui suit supposons que

$$(6) \quad \frac{\partial f_i}{\partial x_j}(t, x) > 0, \quad \forall i \neq j, \quad \forall (t, x)$$

et soit $x_0(t)$ la solution ω -périodique du système (2). Alors on peut formuler le

Théorème 2. Si on a $\frac{\partial f_i}{\partial x_j}(t, x) > 0, \quad \forall i \neq j, \quad \forall (t, x)$, et la solution banale du système en variations est uniformément asymptotiquement stable, alors $\exists \varepsilon_0 > 0$, tel que pour $|\varepsilon| < \varepsilon_0$ il existe une solution $x(t, \varepsilon)$, ω -périodique et de plus $\lim_{\varepsilon \rightarrow 0} x(t, \varepsilon) = x_0(t)$.

Démonstration. Posons $g_{ij}(t) = \frac{\partial f_i}{\partial x_j}(t, x_0(t))$ et alors le système (3) a le système homogène associé de la forme $\dot{y}(t) = g(t)y(t)$, étant justement le système en variations.

Puisque la solution banale de ce système est uniformément asymptotiquement stable il résulte que celle-ci est exponentiellement stable, donc $\exists \alpha, \beta > 0$ tels que $|G(t, s)| \leq \beta e^{-\alpha(t-s)}, \quad \forall t \geq s$.

Par la suite, on a les inégalités

$$\left| \int_{-\infty}^t G(t, s) \cdot 1 \, ds \right| \leq \int_{-\infty}^t |G(t, s)| \, ds \leq \beta \int_{-\infty}^t e^{-\alpha(t-s)} \, ds = \frac{\beta}{\alpha}.$$

Donc la fonction

$$(7) \quad y(t) = \int_{-\infty}^t G(t, s) \cdot 1 \, ds$$

est une solution positive du système (3). La solution (7) est positive puisque $G_{ij}(t, s) \geq 0$. En effet, en comparaisant le système $\dot{y}(t) = g(t)y(t)$ avec lui-même, on a

$$\begin{cases} g_{ii}(t) \leq g_{ii}(t), \\ |g_{ij}(t)| \leq g_{ij}(t), \end{cases} \quad \forall i = 1, \dots, n, \quad i \neq j, \quad \forall t \in \mathbb{R}$$

et donc $|G_{ij}(t, s)| \leq G_{ij}(t, s)$. La fonction (7) est une solution positive et bornée sur $\mathbb{R}$, donc le système en variations n'a pas des solutions ω -périodiques, sauf la solution banale et par conséquent la démonstration est finie.

Si on considère le système quasilinear

$$(8) \quad \dot{x}(t) = A(t)x(t) + f(t) + \varepsilon g(t, x, \varepsilon)$$

on remarque le fait que le système

$$(9) \quad \dot{y}(t) = A(t)y(t)$$

a la solution banale uniformément asymptotiquement stable. Si on suppose que $a_{ij} > 0 \quad \forall i \neq j$, alors le système générateur $\dot{x}(t) = A(t)x(t) + f(t)$ a une solution ω -périodique unique, de la forme : $x_0(t) = \int_{-\infty}^t G(t, s)f(s)ds$. Les hypothèses

nous assurent qu'il existe $\varepsilon_0 > 0$ tel que pour $|\varepsilon| < \varepsilon_0$ le système quasilineaire a une solution ω -périodique, $x(t, \varepsilon)$ ayant la propriété $\lim_{\varepsilon \rightarrow 0} x(t, \varepsilon) = x_0(t)$.

Reçue le 20 Novembre, 1992

Institut Polytechnique, Jassy,
Chaire de Mathématiques

BIBLIOGRAPHIE

1. Halanay A., *Teoria calitativă a ecuațiilor diferențiale*. Ed. Acad. R.S.R., Buc., 1963.
2. Simirad C., Ursescu C., *On Boundedness of the Solutions and Oscillations of some Classes of Differential Systems* (en cours d'apparition).
3. Simirad C., Ursescu C., *Differential Inequalities Concerning Linear Systems* (en cours d'apparition).

ASUPRA SISTEMELOR DE ECUAȚII DIFERENȚIALE CU PARAMETRU MIC

(Rezumat)

Se consideră un sistem diferențial generator cu perturbații ce conțin ca factor un parametru mic, căruia i se asociază un sistem diferențial linear. Ipoteze simple asupra acestui sistem linear asociat asigură existența unei soluții unice pentru sistemul generator, care aproximează soluția sistemului perturbat. Se evită astfel o ipoteză dură, greu de verificat, din metoda parametrului mic, punându-se la îndemână un mijloc practic de analiză calitativă.

PERIODICAL AND OSCILLATORY SOLUTIONS OF AN AGE DEPENDENT POPULATION MODEL

BY

I. ONCIULESCU

1. Introduction. Statement of the Problem. In [2] M. E. Gurtin and R. C. MacCamy have studied a mathematical model of age dependent population dynamics, in which the growth of a certain age population as well as the number of offsprings per unit time, depend essentially on the total population. Starting from this model, our purpose is to study the periodicity of this phenomenon.

We use the following notations:

i) $u(t, a)$ — the number of individuals of age a , at the moment t , of a population P ;

ii) $Du(t, a) = \lim_{h \rightarrow 0} \frac{u(t+h, a+h) - u(t, a)}{h}$;

iii) $\lambda(t, a)$ — the age-specific death coefficient assumed to be negative ($\lambda(t, a) \leq 0$);

iv) $\mu(t, a)$ — the age specific fertility;

v) $\alpha(a)$ — the initial repartition of the age population a ;

vi) $\beta(t)$ — the number of offsprings at the moment t .

More concerning the functions λ , μ , α will be stated in the following. A mathematical model for this population is the following system

[2], [3]:

$$(1) \quad Du(t, a) = \lambda u(t, a), \quad t \geq 0, \quad a \geq 0 \quad (\text{evolution equation}),$$

$$(2) \quad \beta(t) = \int_0^{\infty} \mu(t, \xi) u(t, \xi) d\xi, \quad t \geq 0 \quad (\text{offspring equation}),$$

$$(3) \quad u(0, a) = \alpha(a), \quad a \geq 0 \quad (\text{initial repartition of the population}),$$

$$(4) \quad \beta(0) = \alpha(0) \quad (\text{compatibility equation}).$$

Our aim is to investigate in the present paper some cases when this system has a periodical or oscillatory solution $u(t, a)$ of C^1 class on the domain $\bar{D} = \{(t, a), t \geq 0, a \geq 0\}$ with respect to t .

Concerning the functions λ , μ and α we suppose:

$$(h_1) \quad \lambda \text{ is a negative constant } (\lambda \leq 0);$$

$$(h_2) \quad \alpha(a) = \varphi(a) \exp(\lambda a), \text{ where } \varphi \text{ is a function of } C^1 \text{ class on } [0, \infty);$$

(h₃) $\mu(t, a) = \eta(t)a^n \exp(\mu_0 a)$, $t \geq 0$, $a \geq 0$, where $\mu(t)$ is a C^1 class function on $[0, \infty)$, $\eta(t) \geq 0$, $n \geq 0$ is an integer and μ_0 a real negative constant.

In this paper we shall prove some conditions when the system (1)–(4) has, in case $n=0$, a periodical solution $u(t, a)$ with period $\omega > 0$ with respect to t , and an oscillatory solution $u(t, a)$ with respect to t in the case $\eta(t)$ is a constant, $n > 0$.

2. A Transformation of the System (1)–(4). Let $(u(t, a), \beta(t))$, $(t, a) \in D$, be a solution of C^1 class for the system (1)–(4). If $(t_0, a_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ is a fixed point and if we denote $\bar{u}(s) = u(t_0 + s, a_0 + s)$, $s \geq 0$, then $Du(t_0 + s, a_0 + s) = = \bar{u}'(s)$.

It follows from (1) that $\bar{u}'(s) = \lambda \bar{u}(s)$ and hence $\bar{u}(s) = \bar{u}(0) \exp(\lambda s)$, and

$$(5) \quad u(t_0 + s, a_0 + s) = u(t_0, a_0) \exp(\lambda s).$$

Replacing in (5) $a_0 = 0$, $t_0 = t - a$, $s = a$, when $a \leq t$, we obtain $u(t, a) = = u(t - a, 0) \exp(\lambda a)$, i. e.

$$(6) \quad u(t, a) = \beta(t - a) \exp(\lambda a) \quad \text{if } a \leq t.$$

Replacing in (5) $a_0 = a - t$, $t_0 = 0$, $s = t$, when $a \geq t$, we obtain, taking into account (h₂), $u(t, a) = \varphi(a - t) \exp(\lambda a)$, and hence

$$(7) \quad u(t, a) = \begin{cases} \varphi(a - t) \exp(\lambda a) & \text{if } 0 \leq t \leq a, \\ \beta(t - a) \exp(\lambda a) & \text{if } 0 \leq a \leq t. \end{cases}$$

So the system (1)–(4) leads to the system (7), (2), (3), (4).

3. A Necessary Condition of Periodicity of a Solution of the System (1)–(4). **Proposition.** *If $u = u(t, a)$ from a solution $(u(t, a), \beta(t))$ of the system (1)–(4) is periodical with period $\omega > 0$ with respect to t , then the function φ (from (h₂)) is periodical with period ω .*

Proof. Suppose $u(t, a)$ is periodical with respect to t . From (7) we obtain

$$u(t + \omega, a) = \begin{cases} \varphi(a - t - \omega) \exp(\lambda a) & \text{if } t + \omega \leq a, \\ \beta(t + \omega - a) \exp(\lambda a) & \text{if } t + \omega \geq a. \end{cases}$$

Then, because $u(t, a)$ is periodical with period ω , it follows

$$(8) \quad \varphi(a - t - \omega) = \varphi(a - t), \quad \forall a, t \geq 0, \quad t + \omega \leq a.$$

Take now a number $s \geq 0$. Replacing in (8) $a, t \geq 0$ so that $a - t = s + \omega$, it results $\varphi(s) = \varphi(s + \omega)$, that is the fact that φ is periodical with period ω .

4. A Case When the Solution of System (1)–(4) is Periodical with Respect to t . **Lemma 1.** *Let the system (1)–(4) satisfies the hypotheses (h₁), (h₂) and (h₃) where $n=0$ and*

$$(a_1) \quad \varphi = \varphi(a), \quad a \geq 0 \quad \text{and} \quad \eta = \eta(t), \quad t \geq 0 \quad \text{are periodical with period } \omega > 0;$$

$$(a_2) \quad (\lambda + \mu_0)\omega + \int_0^\omega \eta(s) ds = 0;$$

$$(a_3) \quad \exp \left(- \int_t^{\omega} \eta(s) ds \right) = \frac{1}{I} \exp((\lambda + \mu_0)\omega) \left(\int_0^t \varphi(\omega - s) (\exp(-(\lambda + \mu_0)s)) ds + I \right),$$

$$\forall t \in [0, \omega], \quad I = \int_0^{\omega} \varphi(s) (\exp(\lambda + \mu_0)s) ds.$$

Then, if $(u(t, a), \beta(t))$ is a solution of the system (1)–(4) we have

$$(8) \quad \beta(t) = I \eta(t) \exp \left[(\lambda + \mu_0)t + \int_0^t \eta(s) ds \right], \quad \forall t \geq 0,$$

$$(9) \quad \beta(t + \omega) = \beta(t), \quad \forall t \geq 0,$$

$$(10) \quad \beta(\omega - t) = \omega(t), \quad \forall t \in [0, \omega].$$

P r o o f. Let $(u(t, a), \beta(t))$ be a solution of the system (1)–(4). Then, taking into account (2), (7) and (h_3) we can write

$$\begin{aligned} \beta(t) &= \int_0^{\omega} \eta(t) (\exp(\mu_0 \xi)) u(t, \xi) d\xi = \eta(t) \left[\int_0^t (\exp(\lambda + \mu_0)\xi) \beta(t - \xi) d\xi + \right. \\ &\left. + \int_t^{\omega} \exp(\lambda + \mu_0)\xi \varphi(\xi - t) d\xi \right] = \eta(t) (\exp(\lambda + \mu_0)t) \left[\int_0^t (\exp(-(\lambda + \mu_0)\xi) \beta(\xi) d\xi + I \right], \end{aligned}$$

and then

$$(11) \quad (\exp[-(\lambda + \mu_0)t] \beta(t) = \eta(t) \int_0^t \exp[-(\lambda + \mu_0)\xi] \beta(\xi) d\xi + \eta(t) I.$$

We denote $v(t) = \int_0^t (\exp(-(\lambda + \mu_0)\xi)) \beta(\xi) d\xi.$

Then, from (11) it results $v'(t) = \eta(t)v(t) + \eta(t)I$, which leads to

$$v(t) = I \left[\exp \left(\int_0^t \eta(s) ds \right) - I \right]$$

and then (8).

Further, periodicity of β , i. e. $\beta(t + \omega) - \beta(t) = 0$, leads using the hypotheses (a_1) , (a_2) , to

$$\begin{aligned} \beta(t + \varphi) - \beta(t) &= \\ &= I \left\{ \eta(t + \omega) \left[\exp \left(\int_0^{t+\omega} (\eta(s) ds + (\lambda + \mu_0)(t + \omega)) \right) \right] - \eta(t) \exp \left(\int_0^t \eta(s) ds + (\lambda + \mu_0)t \right) \right\} = \end{aligned}$$

$$\begin{aligned}
 &= I\eta(t) \exp\left(\int_0^t \eta(s) ds + (\lambda + \mu_0)t\right) \left[\exp\left(\int_t^{t+\omega} \eta(s) ds + (\lambda + \mu_0)\omega\right) - 1 \right] = \\
 &= I\eta(t) \left[\exp\left(\int_0^t \eta(s) ds + (\lambda + \mu_0)t\right) \right] \left[\exp\left(\int_0^\omega \eta(s) ds + (\lambda + \mu_0)\omega\right) - 1 \right],
 \end{aligned}$$

which vanishes, as a conclusion of (9).

Now we check the equality (10): taking into account (9) and (a_2) we have

$$\begin{aligned}
 \beta(\omega - t) &= I\eta(\omega - t) \exp\left((\lambda + \mu_0)(\omega - t) + \int_0^{\omega - t} \eta(s) ds\right) = \\
 &= I\eta(\omega - t) (\exp(-(\lambda + \mu_0)t)) \left[\exp\left(-\int_{\omega - t}^\omega \eta(s) ds\right) \right].
 \end{aligned}$$

Because of the condition (a_3) , first we have

$$\eta(t) \exp\left(-\int_t^\omega \eta(s) ds\right) = \frac{1}{I} (\exp(\lambda + \mu_0)\omega) \varphi(\omega - t) \exp(-(\lambda + \mu_0)t),$$

and then

$$\eta(\omega - t) \exp\left(-\int_{\omega - t}^\omega \eta(s) ds\right) = \frac{1}{I} \exp[(\lambda + \mu_0)\omega - (\lambda + \mu_0)(\omega - t)] = \frac{1}{I} \varphi(t) \exp(\lambda + \mu_0)t,$$

$$\forall t \in [0, \omega].$$

Replacing in (11) we obtain the equality (10).

Remark. As a consequence of (9) and (10) we have

$$(12) \quad \beta(0) = \varphi(0)$$

and

$$(13) \quad \beta'(0) = -\varphi'(0).$$

Theorem 1. Let the system (1)–(4) satisfies the hypotheses (h_1) , (h_2) , (h_3) (where $n=0$) and (a_1) , (a_2) , (a_3) .

Then, this system has a unique solution of C^1 class, periodical with period ω with respect to t .

Proof. If $(u(t, a), \beta(t))$ is a solution of the system (1)–(4), then $u(t, a)$ has the form (7), in which $\beta(t)$ is given by (8) (Lemma 1). Because of the equalities (12), (13), the so determined $(u(t, a), \beta(t))$ represents a solution of C^1 class on the domain $D = \{(t, a), t \geq 0, a \geq 0\}$, for the system (1)–(4).

Next we check the periodicity of the solution $u(t, a)$ determined at (8): if $\omega < a$ we can write, using (9), (10)

$$u(t + \omega, a) = \begin{cases} \varphi(a - t - \omega) \exp(\lambda a) & \text{if } 0 \leq t + \omega \leq a \\ \beta(t + \omega - a) \exp(\lambda a) & \text{if } a \leq t + \omega \end{cases} =$$

$$= \begin{cases} \varphi(a-t) \exp(\lambda a) & \text{if } 0 \leq t \leq a - \omega \\ \beta(\omega - (a-t)) \exp(\lambda a) & \text{if } a - t \leq t \leq a \\ \beta(t-a) \exp(\lambda a) & \text{if } a \leq t \end{cases} =$$

$$= \begin{cases} \varphi(a-t) \exp(\lambda a) & \text{if } 0 \leq t \leq a \\ \beta(t-a) \exp(\lambda a) & \text{if } t \geq a \end{cases} = u(t, a).$$

If $a < \omega$, we can write, using (10),

$$u(t+\omega, a) = \beta(t+\omega-a) \exp(\lambda a) = \begin{cases} \varphi(a-t) \exp(\lambda a) & \text{if } 0 \leq t \leq a \\ \beta(t-a) \exp(\lambda a) & \text{if } t \geq a \end{cases} = u(t, a).$$

5. The Case $n \neq 0$ when the Solution of the System (1)–(4) is Oscillatory with respect to t . Lemma 2. Let the system (1)–(4) satisfies the hypotheses (h_1) , (h_2) and (h_3) in which $\eta(t) = \eta_0^{n+1}/n!$, η_0 a constant, $n \geq 1$ an integer. Then, if $(u(t, a), \beta(t))$ is a solution of the system (1)–(4) we have

$$(14) \quad \beta(t) = \frac{\eta_0^{n+1}}{(n+1)!} (\exp((\lambda + \mu_0)t)) \left[2P_n(0) \sum_{k=1}^p \left(\exp \left(\eta_0 \left(\cos \frac{2k\pi}{2p+1} \right) t \right) \right) \cdot \right. \\ \cdot \left(\cos \eta_0 \left(\sin \frac{2k\pi}{2p+1} \right) t \right) + 2 \sum_{k=1}^p \int_0^t P'_n(t-\tau) \left(\exp \left(\eta_0 \left(\cos \frac{2k\pi}{2p+1} \right) \tau \right) \right) \cdot \\ \cdot \left(\cos \eta_0 \left(\sin \frac{2k\pi}{2p+1} \right) \tau \right) d\tau + P_n(0) \exp(\eta_0 t) + \int_0^t P'_n(t-\tau) (\exp(\eta_0 \tau)) d\tau \Big]$$

when $n=2p$, and

$$(14') \quad \beta(t) = \frac{\eta_0^{n+1}}{(n+1)!} (\exp((\lambda + \mu_0)t)) \left[2P_n(0) \sum_{k=1}^{p-1} \left(\exp \left(\eta_0 \left(\cos \frac{k\pi}{p} \right) t \right) \right) \cdot \right. \\ \cdot \left(\cos \eta_0 \left(\sin \frac{k\pi}{p} \right) t \right) + 2 \sum_{k=1}^{p-1} \int_0^t P'_n(t-\tau) \left(\exp \left(\eta_0 \left(\cos \frac{k\pi}{p} \right) \tau \right) \right) \left(\cos \eta_0 \left(\sin \frac{k\pi}{p} \right) \tau \right) d\tau + \\ + P_n(0) (\exp(\eta_0 t) + \exp(-\eta_0 t)) + \int_0^t P'_n(t-\tau) (\exp(\eta_0 \tau) + \exp(-\eta_0 \tau)) d\tau \Big]$$

when $n=2p-1$, where

$$P_n(t) = \int_0^\infty (\xi + t)^n \exp((\lambda + \mu_0)\xi) d\xi.$$

Proof. If $(u(t, a), \beta(t))$ is a solution of the system (1)–(4), taking into account (2), (7) and (h_3) in which $\eta(t) = \eta_0^{n+1}/n!$, we obtain, analogous as in the proof of the Lemma 1, that

$$(15) \quad (\exp(-(\lambda + \mu_0)t)) \beta(t) = \frac{\eta_0^{n+1}}{n!} \int_0^t (t-\xi)^n (\exp(-(\lambda + \mu_0)\xi)) \beta(\xi) d\xi + \frac{\eta_0^{n+1}}{n!} P_n(t).$$

Denoting

$$v(t) = \int_0^t \exp(-(\lambda + \mu_0)\xi) \beta(\xi) d\xi,$$

(15) becomes

$$(16) \quad v'(t) = \frac{\gamma_0^{n+1}}{n!} \int_0^t (t-\xi)^n v'(\xi) d\xi + \frac{\gamma_0^{n+1}}{n!} P_n(t).$$

Let $V(p)$, $F(p)$ be the Laplace transforms of $v(t)$ and $P_n(t)$ respectively. Then, from (16) it results

$$V(p) = \frac{\gamma_0^{n+1}}{(n+1)!} \frac{(n+1)P^n}{P^{n+1} - \gamma_0^{n+1}} F(p),$$

$$v(t) = \frac{\gamma_0^{n+1}}{(n+1)!} \sum_{k=0}^n \int_0^t P_n(t-\tau) \exp(p_k \tau) d\tau,$$

where

$$p_k = \gamma_0 \left(\cos \frac{2k\pi}{n+1} + i \sin \frac{2k\pi}{n+1} \right), \quad k = \overline{0, n}.$$

Replacing the so determined $v(t)$ in the equality $\beta(t) = v'(t) \exp((\lambda + \mu_0)t)$, it results (14) or (14').

Theorem 2. Suppose the system (1)–(4) satisfies the hypotheses (h_1) , (h_2) and (h_3) wherein $\gamma(t) = \gamma_0^{n+1}/n!$, γ_0 being a constant; if

$$(b_1) \quad \frac{\gamma_0^{n+1}}{(n+1)!} \int_0^\infty \xi^n (\exp((\lambda + \mu_0)\xi)) \varphi(\xi) d\xi = \varphi(0),$$

$$(b_2) \quad \frac{\gamma_0^{n+1}}{n!} \left[(\lambda + \mu_0) \int_0^\infty \xi^n (\exp((\lambda + \mu_0)\xi)) \varphi(\xi) d\xi + n \int_0^\infty \xi^{n-1} (\exp((\lambda + \mu_0)\xi)) \varphi(\xi) d\xi \right] = -\varphi'(0),$$

$$(b_3) \quad \lambda + \mu_0 + \gamma_0 < 0,$$

then the system (1)–(4) has a unique solution $(u(t, a), \beta(t))$ of class C^1 , $u(t, a)$ with the form (7) in which $\beta(t)$ has the form (14) or (14') and $\lim_{t \rightarrow \infty} u(t, a) = 0$ for an age a fixed.

Proof. If $(u(t, a), \beta(t))$ is a solution of the system (1)–(4), then $u(t, a)$ has the form (7) and $\beta(t)$ has the form (14) or (14') (Lemma 2), so that the solution of the system is unique. As a consequence of the conditions (b_1) and (b_2) , we have the equalities (13) and (14). It results that $(u(t, a), \beta(t))$ so determined is a solution of C^1 class for the system (1)–(4).

If (b_3) is satisfied, we have $\lim_{t \rightarrow \infty} u(t, a) = 0$ for every $a \geq 0$ fixed.

Received October 3, 1991

Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Ditkine V., Proudnikov A., *Calcul Opérationnel*. Éd. Mir, Moscou, 1979.
2. Gurtin M., Mc Camy R. C., *Non Linear Age Dependent Population Dynamics*. Arch. Rational Mech. Anal., **54**, 281–300 (1974).
3. Hoppensteadt F., *Mathematical Theories of Population: Demographics, Genetics and Epidemics*. Regional Conference, Series in Applied Mathematics, **80**, SIAM, 1975.
4. Volterra V., *Leçons sur la Théorie Mathématique de la lutte pour la vie*. Paris, Gauthier-Villars, 1931.

SOLUȚII PERIODICE ȘI OSCILATORII ALE UNUI MODEL DE POPULAȚIE DEPENDENT DE VÂRSTĂ

(Rezumat)

Pentru sistemul (1)–(4), satisfăcând ipotezele (h_1) , (h_2) , (h_3) , care reprezintă un model pentru o populație structurată pe vârste, se determină condiții când acest sistem admite soluții periodice în raport cu timpul t (Teorema 1) și condiții când sistemul admite soluții oscilatorii în raport cu timpul t (Teorema 2).

THE UNIVERSITY OF CHICAGO PRESS

1914

THE UNIVERSITY OF CHICAGO PRESS

1914

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

CONTINUOUS SELECTIONS OF MULTI-VALUED MAPS WITH NON CONVEX RIGHT-HAND SIDE AND THE GOURSAT PROBLEM FOR THE MULTI-VALUED HYPERBOLIC EQUATION

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$$

BY

GEORGETA TEODORU

1. Introduction. In this paper we are concerned with the Goursat problem for the multi-valued hyperbolic equation $\partial^2 z / \partial x \partial y \in F(x, y, z)$, where F is a multi-valued map, defined in a suitable subset of $\mathbb{R}^{n+2}$, with values that are nonempty compact but not necessarily convex subsets of $\mathbb{R}^n$. The Goursat problem is defined by analogy with the Goursat problem for quasilinear hyperbolic equations [7] in [8], [9], where F is a multi-valued map defined on a subset of $\mathbb{R}^{n+2}$ and valued in the set of compact convex nonempty subsets of $\mathbb{R}^n$, satisfying conditions of Caratheodory type. Using the Fixed Point Theorem of Kakutani-Ky Fan one proves that the problem above has at least a solution.

In this Note one proves an existence theorem of a continuous selection for each of the maps $(x, y) \rightarrow F(x, y, z(x, y))$ relative to a given family of continuous maps $(x, y) \rightarrow z(x, y)$, as in [1], [2], [10], [11], [12] and using the Schauder Fixed Point Theorem one obtains the existence of a solution of the Goursat problem.

2. Continuous Approximate Selections. Let the multi-valued map $F: A \rightarrow \text{comp } X$, where $A \subset \mathbb{R}^{n+2}$, $A = P \times B$, $P = [-\alpha, \alpha] \times [-\beta, \beta] \subset \mathbb{R}^2$ given by the hypothesis (H_0) , $B \subset \mathbb{R}^n$ is the closed ball centered in origin with radius $c = M_1 + Mab$, M_1 given by (2), M given by (3), $X \subset \mathbb{R}^n$ is the closed ball centered in origin with radius M , X being a compact metric space with the metric d induced on X by the norm defined on $\mathbb{R}^n$.

Let H be the Hausdorff-Pompeiu metric [6] on $\text{comp } X$ induced by d . Then $\text{comp } X$ is a compact metric space with respect to the metric H .

Let $C(P; \mathbb{R}^n)$ be the Banach space of continuous functions defined on P and valued in $\mathbb{R}^n$ and $\mathcal{L}^1(P; \mathbb{R}^n)$ the Banach space of equivalence classes of Lebesgue integrable functions defined on P and valued in $\mathbb{R}^n$.

Let the following hypotheses be satisfied :

(H_0) Let α, β, a, b positive numbers and $y = g(x) : [0, a] \rightarrow \mathbb{R}$, $x = h(y) : [0, b] \rightarrow \mathbb{R}$, be nondecreasing functions of class C^1 such that $g(0) = h(0) = 0$, $0 \leq g(x) \leq b$, $0 \leq h(y) \leq a$.

Let $P = [-\alpha, \alpha] \times [-\beta, \beta]$, $\tilde{D} = [0, a] \times [0, b]$,

$$D = \{(s, t) | h(t) < s \leq a, g(s) < t \leq b\},$$

$$D_{xy} = \{(s, t) | h(t) < s \leq x, g(s) < t \leq y\},$$

for $x \in [0, a]$ and $y \in [0, b]$ and $G = P - D$.

Let $\varphi : P \rightarrow \mathbb{R}^n$ an absolutely continuous function,
 (H₁) $\varphi \in C^*(P; \mathbb{R}^n)$, ([3], § 565–§ 568), and $\lambda : \tilde{D} \rightarrow \mathbb{R}^n$ be the bounded function given by the relation

$$(1) \quad \lambda(x, y) = \varphi(0, 0) + \int_0^x \varphi'_x(s, g(s)) ds + \int_0^y \varphi'_y(h(t), t) dt;$$

there exists $M_1 > 0$ such that

$$(2) \quad \|\lambda(x, y)\| \leq M_1, \quad (x, y) \in \tilde{D}.$$

It follows that λ is absolutely continuous; $\lambda \in C^*(\tilde{D}; \mathbb{R}^n)$ ([3], § 565–§ 568).

Let K be the set of absolutely continuous functions $z : P \rightarrow \mathbb{R}^n$, $z \in C^*(P; \mathbb{R}^n)$ [3] satisfying the conditions (2), (3), (4), where

$$(3) \quad \left\| \frac{\partial^2 z(x, y)}{\partial x \partial y} \right\| \leq M, \quad \text{a.e. } (x, y) \in D,$$

and

$$(4) \quad z(x, y) = \varphi(x, y), \quad (x, y) \in G = P - D.$$

Then, the following two propositions hold :

Proposition 1. K is a nonempty convex and compact subset of the Banach space $C(P; \mathbb{R}^n)$.

Proof. The relation $z \in K$ implies $z \in C(P; \mathbb{R}^n)$. We observe that $\partial^2 z / \partial x \partial y$ exists a. e. $(x, y) \in P$, as $z \in C^*(P; \mathbb{R}^n)$ [3].

Integrating $\partial^2 z(x, y) / \partial x \partial y$ over $\overline{D}_{xy}$ and using (4) one obtains

$$(5) \quad z(x, y) = \varphi(0, 0) + \int_0^x \varphi'_x(s, g(s)) ds + \int_0^y \varphi'_y(h(t), t) dt + \iint_{\overline{D}_{xy}} \frac{\partial^2 z(s, t)}{\partial s \partial t} ds dt,$$

or using (1)

$$(5') \quad z(x, y) = \lambda(x, y) + \iint_{\overline{D}_{xy}} \frac{\partial^2 z(s, t)}{\partial s \partial t} ds dt, \quad (x, y) \in D.$$

The compactness of K follows using (5) or (5') and the Arzelà-Ascoli Theorem. The convexity of K is obvious.

Remark. The relation $z \in K$ implies $(x, y, z(x, y)) \in A$ for each $(x, y) \in P$. Because each $z \in K$ generates a multi-valued map $(x, y) \rightarrow F(x, y, z(x, y))$ from P to $\text{comp } X$, we shall denote this map by $\tilde{F}(z)$,

$$(6) \quad \tilde{F}(z)(x, y) = F(x, y, z(x, y)), \quad (x, y) \in P.$$

Proposition 2. Let $F: A \rightarrow \text{comp } X$ be a multi-valued continuous map. Then, for each $\varepsilon > 0$, there exists a continuous function $f: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ such that for each $z \in K$ have

$$(7) \quad d(f(z)(x, y), \tilde{F}(z)(x, y)) < \varepsilon, \quad \text{a.e.} \quad (x, y) \in \tilde{D}.$$

Proof. The proof is analogous to that given in [1], [2], [10], [11], [12], and is based on the construction of the function f by means of the continuous partition of the unity.

Let $\varepsilon > 0$ be given. In view of the fact that F is continuous on A and A is compact, F is uniformly continuous on A and there is $\Delta > 0$ such that

$$H(F(x, y, z), F(\xi, \eta, \tilde{z})) < \varepsilon,$$

for any two points $(x, y, z), (\xi, \eta, \tilde{z})$ in A with $\|(x, y) - (\xi, \eta)\| < \Delta$, $\|z - \tilde{z}\| < \Delta$.

Let $\{\mathcal{U}_k\}_{1 \leq k \leq N}$ be a finite open cover of K such that $\text{diam } \mathcal{U}_k < \Delta$ for any $k = \overline{1, N}$. Let $\{p_k\}_{1 \leq k \leq N}$ be the continuous partition of unity subordinate to $\{\mathcal{U}_k\}_{1 \leq k \leq N}$; select for each k a point $z_k \in \mathcal{U}_k$ and let $\{v_k\}_{1 \leq k \leq N}$ be a sequence of Lebesgue measurable functions $v_k: \tilde{D} \rightarrow \mathbb{R}^n$ such that, for every k , $v_k(x, y) \in \tilde{F}(z_k)(x, y)$ a. e. $(x, y) \in \tilde{D}$. Such function v_k exists because each $\tilde{F}(z_k)$ is continuous and measurable in $\tilde{D}$ [4]; $v_k \in \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ for every k . We can take $N = N_1 N_2$. Denote $\mathcal{U}_k = \mathcal{U}_{ij}$, $v_k = v_{ij}$, $z_k = z_{ij} \in \mathcal{U}_{ij}$, $p_k(z) = p_{ij}(z)$ and suppose

$$p_{ij}(z) = q_i(z)r_j(z), \quad i = \overline{1, N_1}, \quad j = \overline{1, N_2}.$$

The functions $p_{ij}: K \rightarrow \mathbb{R}$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$ satisfy the properties

- a) $0 \leq p_{ij}(z) \leq 1$, for $z \in K$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$,
- b) $p_{ij}(z) = 0$ if $z \notin \mathcal{U}_{ij}$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$,
- c) $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} p_{ij}(z) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} q_i(z)r_j(z) = 1$, for $z \in K$.

For each $z \in K$ define the continuous functions $\lambda_{ij}: K \rightarrow \mathbb{R}$,

$$\lambda_{ij}(z) = x_i(z)y_j(z), \quad i = \overline{1, N_1}, \quad j = \overline{1, N_2},$$

where

$$\begin{cases} x_0(z) = 0, \\ x_i(z) = x_{i-1}(z) + a q_i(z) \sum_{j=1}^{N_2} r_j(z), \quad i = \overline{1, N_1}, \end{cases}$$

and

$$\begin{cases} y_0(z) = 0, \\ y_j(z) = y_{j-1}(z) + b r_j(z) \sum_{i=1}^{N_1} q_i(z), \quad j = \overline{1, N_2}. \end{cases}$$

For each $z \in K$ define the rectangles

$$D_{ij}(z) = [x_{i-1}(z), x_i(z)] \times [y_{j-1}(z), y_j(z)], \quad i = \overline{1, N_1}, \quad j = \overline{1, N_2},$$

which constitutes a partition of $\tilde{D}$ excepting lines $x=a$ and $y=b$.

We construct the desired function $f: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$,

$$(8) \quad \begin{cases} f(z)(x, y) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \chi[D_{ij}(z)](x, y) v_{ij}(x, y), & 0 \leq x < a, 0 \leq y < b, \\ f(z)(a, y) = v_{l, N_2}(a, y), & l = \min \{i \geq 1; x_i(z) = a\}, \\ f(z)(x, b) = v_{N_1, p}(x, b), & p = \min \{j \geq 1; y_j(z) = b\}, \end{cases}$$

where $\chi[D_{ij}(z)]$ is the characteristic function of $D_{ij}(z)$.

Obviously, f maps K into $\mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$. Moreover, for a given $z \in K$ and any fixed $(x, y) \in \tilde{D}$, there exists a unique (i, j) such that $(x, y) \in D_{ij}(z)$ and this implies $z \in \mathcal{N}_{ij}$. Thus, $f(z)(x, y) = v_{ij}(x, y)$ and $\|z(x, y) - z_{ij}(x, y)\| < \Delta$ so that

$$(9) \quad \begin{aligned} d(f(z)(x, y), \tilde{F}(z)(x, y)) &\leq d(v_{ij}(x, y), \tilde{F}(z_{ij})(x, y)) + \\ &+ H(\tilde{F}(z_{ij})(x, y), \tilde{F}(z)(x, y)) \leq d(v_{ij}(x, y), \tilde{F}(z_{ij})(x, y)) + \varepsilon. \end{aligned}$$

It follows that, for each $z \in K$, $d(f(z)(x, y), \tilde{F}(z)(x, y)) < \varepsilon$, a. e. $(x, y) \in \tilde{D}$, therefore (7) holds. We show that f is continuous on K . Then, for any points z, w in K and any $(x, y) \in \tilde{D}$, $0 \leq x < a$, $0 \leq y < b$,

$$(10) \quad \|f(z)(x, y) - f(w)(x, y)\| \leq \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \chi[D_{ij}(z) \Delta D_{ij}(w)](x, y) \|v_{ij}(x, y)\|,$$

where $D_{ij}(z) \Delta D_{ij}(w) = (D_{ij}(z) - D_{ij}(w)) \cup (D_{ij}(w) - D_{ij}(z))$.

Since K is compact, $\{\lambda_{ij}\} \ i = \overline{1, N_1}, j = \overline{1, N_2}$ is a uniformly equicontinuous family of real valued functions. Thus, for every $\eta > 0$, there exists a $\delta > 0$ such that, for any $z \in K$, $w \in K$ satisfying $\|z(x, y) - w(x, y)\| < \delta$ at every $(x, y) \in \tilde{D}$,

$$|\lambda_{ij}(z) - \lambda_{ij}(w)| < \frac{\eta}{2MN},$$

and hence

$$\mu(D_{ij}(z) \Delta D_{ij}(w)) < \frac{\eta}{MN},$$

so that (10) implies

$$(11) \quad \|f(z) - f(w)\|_{\mathcal{L}^1} = \iint_{\tilde{D}} \|f(z)(x, y) - f(w)(x, y)\| dx dy < \eta.$$

Therefore $f: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ is uniformly continuous.

3. Continuous Selections. By analogy with [1], [2], [10], [11], [12], one proves the following existence theorem of a continuous selection for multi-valued map $\tilde{F}(z)$,

Theorem 1. *If $F: A \rightarrow \text{comp } X$ is a multi-valued continuous map, then there exists a continuous function $f: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ such that for any $z \in K$, $f(z)(x, y) \in \tilde{F}(z)(x, y)$, a. e. $(x, y) \in \tilde{D}$ that is $f(z)$ is a continuous selection for $\tilde{F}(z)$, given by (6).*

Proof. Define a sequence of continuous functions $f^n: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$, $n \in \mathbb{N}$, submitted to the following conditions

$$1) \quad d(f^n(z)(x, y), \tilde{F}(z)(x, y)) < \frac{1}{2^{n+1}}, \quad \text{a. e. } (x, y) \in \tilde{D},$$

$$2) \quad \mu \left\{ (x, y) \in \tilde{D} \mid \|f^n(z)(x, y) - f^{n-1}(z)(x, y)\| \geq \frac{1}{2^{n-1}} \right\} < \frac{1}{2^n}.$$

The condition 2) shows that for each $z \in K$, the sequence $\{f^n(z)\}_{n \in \mathbb{N}}$ converges, in the norm of $\mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$, to an element $f(z)$ and the convergence is uniform on K , because the condition 2) is satisfied uniformly for any $z \in K$. Using the Lebesgue Dominated Convergence Theorem it follows that $f(z) \in \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ for each $z \in K$. By continuity of the functions f^n , $n \in \mathbb{N}$, it follows that $f: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ is continuous. Therefore, for any $z \in K$, there exists a measurable function $f(z): \tilde{D} \rightarrow X$ such that the sequence $\{f^n(z)\}_{n \in \mathbb{N}}$ converges to $f(z)$ a. e. in measure, and a subsequence of $\{f^n(z)\}_{n \in \mathbb{N}}$ that converges to $f(z)$ a. e. on $\tilde{D}$. Then, from the condition 1), it follows that for each $z \in K$ we have $f(z)(x, y) \in \tilde{F}(z)(x, y)$, a. e. $(x, y) \in \tilde{D}$, because $\tilde{F}(z)(x, y)$ is closed for any $z \in K$.

The sequence $\{f^n(z)\}_{n \in \mathbb{N}}$ is obtained by induction. From Proposition 2 it follows that there exists a continuous function $f^0: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ such that for any $z \in K$

$$(12) \quad d(f^0(z)(x, y), \tilde{F}(z)(x, y)) < \frac{1}{2}, \quad \text{a.e. } (x, y) \in \tilde{D}.$$

Also, from Proposition 2 and the continuity of F on $\tilde{D} \times B$ there exists $\Delta_1 > 0$ such that

$$(13) \quad H(F(x, y, z), F(\xi, \eta, \tilde{z})) < \frac{1}{4}$$

for each $(x, y, z), (\xi, \eta, \tilde{z})$ in A with $\|(x, y) - (\xi, \eta)\| < \Delta_1$, $\|z - \tilde{z}\| < \Delta_1$ and

$$(14) \quad \mu(\{(x, y) \in \tilde{D} \mid \|f^0(z)(x, y) - f^0(\tilde{z})(x, y)\| > 0\}) < \frac{1}{2}$$

for each $z \in K, \tilde{z} \in K$ satisfying $\|z(x, y) - \tilde{z}(x, y)\| < \Delta_1$ for any $(x, y) \in \tilde{D}$.

By analogy with the Proposition 2, let $\{\mathcal{U}_k^1\}_{1 \leq k \leq N(1)}$ be an open finite cover of K , such that $\text{diam } \mathcal{U}_k^1 < \Delta_1$, for any k ; let $\{p_k^1\}_{1 \leq k \leq N(1)}$ be the continuous partition of unity subordinate to $\{\mathcal{U}_k^1\}_{1 \leq k \leq N(1)}$; we select for each k a point $z_k^1 \in \mathcal{U}_k^1$ and a Lebesgue measurable function $v_k^1: \tilde{D} \rightarrow \mathbb{R}^n$ such that

$$v_k^1(x, y) \in \tilde{F}(z_k^1)(x, y), \quad \text{a.e. } (x, y) \in \tilde{D},$$

and

$$(15) \quad \|v_k^1(x, y) - f^0(z_k^1)(x, y)\| = d(f^0(z_k^1)(x, y), \tilde{F}(z_k^1)(x, y)).$$

This is possible because each $\tilde{F}(z_k^1)$ is continuous hence measurable in $\tilde{D}$ [5]. By analogy with the Proposition 2, consider $N(1) = N_1(1)N_2(1)$ and denote $\mathcal{U}_k^1 = \mathcal{U}_{ij}^1$, $v_k^1 = v_{ij}^1$, $z_k^1 = z_{ij}^1 \in \mathcal{U}_{ij}^1$ and $p_k^1(z) = p_{ij}^1(z) = q_i^1(z)r_j^1(z)$, $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$.

The continuous partition of unity, $\{p_{ij}^1(z)\}$, $p_{ij}^1: K \rightarrow \mathbb{R}$ satisfies:

- a) $0 \leq p_{ij}^1(z) \leq 1$ for all $z \in K$, $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$,
- b) $p_{ij}^1(z) = 0$ if $z \notin \mathcal{U}_{ij}^1$, $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$,
- c) $\sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} p_{ij}^1(z) = \sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} q_i^1(z)r_j^1(z) = 1$ for all $z \in K$.

For each $z \in K$, define the continuous functions $\lambda_{ij}^1: K \rightarrow \mathbb{R}$

$$\lambda_{ij}^1(z) = x_i^1(z)y_j^1(z), \quad i = \overline{1, N_1(1)}, j = \overline{1, N_2(1)},$$

with

$$\begin{cases} x_i^1(z) = 0, \\ x_i^1(z) = x_{i-1}^1(z) + a q_i^1(z) \sum_{j=1}^{N_2(1)} r_j^1(z), \quad i = \overline{1, N_1(1)}, \end{cases}$$

and

$$\begin{cases} y_0^1(z) = 0, \\ y_j^1(z) = y_{j-1}^1(z) + b r_j^1(z) \sum_{i=1}^{N_1(1)} q_i^1(z), \quad j = \overline{1, N_2(1)}. \end{cases}$$

For each $z \in K$ consider the rectangles

$$D_{ij}^1(z) = [x_{i-1}^1(z), x_i^1(z)] \times [y_{j-1}^1(z), y_j^1(z)], \quad i = \overline{1, N_1(1)}, j = \overline{1, N_2(1)},$$

establishing a partition of $\tilde{D}$, except for the lines $x=a$ and $y=b$.

Define the function $f^1: K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$,

$$(16) \quad \begin{cases} f^1(z)(x, y) = \sum_{i=1}^{N_1(1)} \sum_{j=1}^{N_2(1)} \chi[D_{ij}^1(z)](x, y) v_{ij}^1(x, y), & 0 \leq x < a, 0 \leq y < b, \\ f^1(z)(a, y) = v_{N_1(1)}^1(a, y), & l = \min\{i \geq 1; x_i^1(z) = a\}, \\ f^1(z)(x, b) = v_{N_2(1)}^1(x, b), & p = \min\{j \geq 1; y_j^1(z) = b\}. \end{cases}$$

The function f^1 is continuous (see the proof of continuity of f in the Proposition 2). To verify the conditions 1), 2) suppose that $z \in K$ is given, and $(x, y) \in \tilde{D}$ fixed ($0 \leq x < a, 0 \leq y < b$).

Then $(x, y) \in D_{ij}^1(z)$ for a unique pair of indices (i, j) , therefore $p_{ij}^1(z) > 0$. Then

$$(17) \quad f^1(z)(x, y) = v_{ij}^1(x, y)$$

and $\|z(x, y) - z_{ij}^1(x, y)\| < \Delta_1$ such that

$$(18) \quad \begin{aligned} d(f^1(z)(x, y), \tilde{F}(z)(x, y)) &= d(v_{ij}^1(x, y), \tilde{F}(z)(x, y)) \leq \\ &\leq d(v_{ij}^1(x, y), \tilde{F}(z_{ij}^1)(x, y)) + H(\tilde{F}(z_{ij}^1)(x, y), \tilde{F}(z)(x, y)) < \frac{1}{2^2}, \end{aligned}$$

a. e. $(x, y) \in \tilde{D}$, that is the condition 1) holds for $n=1$.

Moreover, using (12), (15) and (17), it follows

$$(19) \quad \begin{aligned} \|f^1(z)(x, y) - f^0(z)(x, y)\| &\leq \|v_{ij}^1(x, y) - f^0(z_{ij}^1)(x, y)\| + \|f^0(z_{ij}^1)(x, y) - \\ &- f^0(z)(x, y)\| = d(f^0(z_{ij}^1)(x, y), \tilde{F}(z_{ij}^1)(x, y)) + \|f^0(z_{ij}^1)(x, y) - \\ &- f^0(z)(x, y)\| < \frac{1}{2} + \|f^0(z_{ij}^1)(x, y) - f^0(z)(x, y)\| \end{aligned}$$

that implies

$$(20) \quad \begin{aligned} \mu(\{x, y \in \tilde{D} \mid \|f^1(z)(x, y) - f^0(z)(x, y)\| > 1\}) &\leq \\ &\leq \mu(\{x, y \in \tilde{D} \mid \|f^0(z_{ij}^1)(x, y) - f^0(z)(x, y)\| > \frac{1}{2}\}) < \frac{1}{2}, \end{aligned}$$

that is the condition 2) holds for $n=1$.

Obviously, a similar construction can be used for $n > 1$, and the theorem is proved.

4. The Goursat Problem. Consider the multi-valued equation

$$(21) \quad \frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \quad (x, y) \in P, \quad z \in B,$$

where $F : P \times B \rightarrow \text{comp } X$.

The Goursat problem is defined in [8], [9] and consists in determination of an absolutely continuous function $z : P \rightarrow \mathbb{R}^n$, $z \in C^*(P; \mathbb{R}^n)$, satisfying the equation (21) for $(x, y) \in \bar{D}$ and the condition (4) for $(x, y) \in G$ in the hypotheses (H_0) , (H_1) .

We state the following

Theorem 2. Let $F : P \times B \rightarrow \text{comp } X$ be a multi-valued map satisfying the hypothesis

$$(H_2) \quad F \text{ is continuous on } A = P \times B.$$

If the hypotheses $(H_0) - (H_2)$ are fulfilled, the Goursat problem (21)+(4) has at least an absolutely continuous solution $\bar{z} : P \rightarrow \mathbb{R}^n$, $\bar{z} \in C^*(P; \mathbb{R}^n)$.

Proof. Using the Theorem 1 it follows that there exists a continuous selection $f : K \rightarrow \mathcal{L}^1(\tilde{D}; \mathbb{R}^n)$ for $\tilde{F}(z)$ given by (6). Define for each $z \in K$, the function $\psi(z) : P \rightarrow \mathbb{R}^n$ by

$$(22) \quad \psi(z)(x, y) = \lambda(x, y) + \iint_{\tilde{D}_{xy}} f(z)(s, t) ds dt, \quad (x, y) \in D,$$

and

$$(23) \quad \psi(z)(x, y) = z(x, y), \quad (x, y) \in G.$$

Then, $\psi(z) \in C^*(P; \mathbb{R}^n)$ for each $z \in K$ [3]. One obtains $\psi(K) \subset K$. Using the Schauder Fixed Point Theorem, it follows that there exists $\bar{z} \in K$ such that $\psi(\bar{z}) = \bar{z}$, that is $\psi(\bar{z})(x, y) = \bar{z}(x, y), (x, y) \in P$. This implies from (22) $\partial^2 \bar{z}(x, y) / \partial x \partial y = f(\bar{z})(x, y) \in \tilde{F}(\bar{z})(x, y) = F(x, y, \bar{z}(x, y))$ for $(x, y) \in \bar{D}$ and $\bar{z}(x, y) = \varphi(x, y)$ for $(x, y) \in G$, therefore (21) and (4) hold for $\bar{z}$, consequently $\bar{z}$ is a solution of the Goursat problem (21)+(4), a. e. $(x, y) \in P$.

Received June 7, 1989

Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Antosiewicz H. A., Cellina A., *Continuous Selections and Differential Relations*. J. Diff. Eq., **19**, 386–398 (1975).
2. Bogatirev A. V., *Continuous Branches of Multi-valued Mappings with Non-convex Right-hand Side* (Russian). Mat. Sb., **120(162)**, 3, 344–353 (1983).
3. Carathéodory C., *Vorlesungen über Reelle Funktionen*. Chelsea Publ. Comp., 1948.
4. Castaing Ch., *Quelques problèmes de mesurabilité liés à la théorie de la commande*. C. R. Acad. Sci., Paris, **262**, 409–414 (1966).
5. Filippov A. F., *Classical Solutions of Differential Equations with Multi-valued Right-hand Side*. SIAM J. Control, **5**, 609–621 (1967).
6. Rus I. A., *Principii și aplicații ale teoriei punctului fix*. Ed. „Dacia”, Cluj-Napoca, 1979.
7. Straburzynski J., *Existence of Solutions of the Goursat Problem for some Functional-differential Equations*. Demonstratio Mathematica, **15**, 4, 883–897 (1982).
8. Teodoru G., *Studiul soluțiilor ecuațiilor de forma $\partial^2 z / \partial x \partial y \in F(x, y, z)$* . Teză de doctorat, Univ. „Al. I. Cuza” Iași, 1984.
9. Teodoru G., *Le problème de Goursat pour une équation aux dérivées partielles multivoque*. Mathematica, **28(51)**, 2, 185–188 (1986).
10. Teodoru G., *Continuous Selections for Multi-valued Functions*. „Itinerant Seminar on Functional Equations, Approximation and Convexity”, Cluj-Napoca „Babeș-Bolyai” University, Research Seminars, Preprint nr. 7, 273–278, 1986.
11. Teodoru G., *Absolutely Continuous Solutions for Cauchy Problem for a Multi-valued Hyperbolic Equation*. Bul. Inst. Polit. Iași, **XXXV(XXXIX)**, 1–2, 47–54 (1989).
12. Teodoru G., *Continuous Selections of Multi-valued Maps with Non-convex Right-hand Side and the Picard Problem for the Multivalued Hyperbolic Equation $\partial^2 z / \partial x \partial y \in F(x, y, z)$* . Studia Univ. „Babeș-Bolyai”, Mathematica, **31**, 3, 58–66 (1986).

SELECȚII CONTINUE PENTRU APLICAȚII MULTIVOCE CU MEMBRUL DREPT
NECONVEX ȘI PROBLEMA LUI GOURSAT PENTRU ECUAȚIA HIPERBOLICĂ
MULTIVOCĂ $\partial^2 z / \partial x \partial y \in F(x, y, z)$

(Rezumat)

Se consideră problema lui Goursat pentru ecuația hiperbolică multivocă $\partial^2 z / \partial x \partial y \in F(x, y, z)$, unde F este o aplicație multivocă continuă definită pe $A \subset \mathbb{R}^{n+2}$ cu valori compacte, dar neconvexe din $\mathbb{R}^n$. Se demonstrează o teoremă de existență a unei selecții continue pentru fiecare din aplicațiile $(x, y) \rightarrow F(x, y, z(x, y))$, relativ la o familie dată de aplicații continue $(x, y) \rightarrow z(x, y) : z \in K$, unde K este o mulțime convexă, compactă, de funcții absolut continue, supuse la anumite condiții. Apoi se definește cu ajutorul acestei selecții un operator pentru care se aplică teorema de punct fix a lui Schauder, punctul fix fiind tocmai soluția problemei lui Goursat.

HEAT CONDUCTION IN AN INFINITE REGION AND HERMITE POLYNOMIALS

BY

S. D. BAJPAI

1. Introduction. The purpose of this paper is to formulate a partial differential equation related to a problem of heat conduction in an infinite region and obtain its series solution in terms of Hermite polynomials.

2. Formulation of the Partial Differential Equation. We formulate the following partial differential equation

$$(2.1) \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + 2kx \frac{\partial u}{\partial x} + 2ku, \quad (-\infty < x < \infty)$$

where $u(x, t)$ tends to zero for large values of t and when $|x| \rightarrow \infty$, provided

$$(2.2) \quad u(x, 0) \equiv u(x).$$

The Eq. (2.1) is related to the problem of heat conduction ([2], p. 29, (1)

$$(2.3) \quad \nabla^2 v - \frac{1}{k} \frac{\partial v}{\partial t} = A(x, y, z),$$

in an infinite region $(-\infty < x < \infty)$, provided $A = -2x \partial u / \partial t - 2u$.

The Eq. (2.1) may be linked to some other problems of applied mathematics and related sciences.

3. Solution of the Partial Differential Equation. The solution of (2.1) to be obtained is

$$(3.1) \quad u(x, t) = \sum_{n=0}^{\infty} C_n e^{-2knt - x^2} H_n(x),$$

where $H_n(x)$ are Hermite polynomials.

Proof. Let us assume that (2.1) has a solution of the form

$$(3.2) \quad u(x, t) = e^{-2knt - x^2} X(x), \quad n = 0, 1, 2, \dots$$

The substitution of (3.2) in (2.1) yields the differential equation

$$(3.3) \quad X''(x) - 2xX'(x) + 2nX(x) = 0,$$

which is Hermite equation ([1], p. 170, (5.16)], with solution $X(x) = H_n(x)$. Therefore, we have

$$(3.4) \quad u(x, t) = e^{-2kn t - x^2} H_n(x).$$

On using the principle of superposition, the solution (3.1) is obtained. Putting $t=0$ in (3.1), we get

$$(3.5) \quad u(x) = \sum_{n=0}^{\infty} C_n e^{-x^2} H_n(x).$$

Multiplying both sides of (3.5) by $H_m(x)$ and integrating with respect to x from $-\infty$ to ∞ , and using the orthogonality property of the orthogonal polynomials ([1], pp. 170–171, (5.17) & (5.22)), we obtain

$$(3.6) \quad C_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} u(x) H_n(x) dx.$$

Note. The n -dimensional partial differential equation analogous to (2.1) can be formulated and solved by following the same procedure as above.

Received June 19, 1991

University of Bahrain,
Department of Mathematics,
Isa Town, Bahrain

REFERENCES

1. Andrews L. C., *Special Functions for Engineers and Applied Mathematics*. Macmillan Publ. Co., New York, 1985.
2. Carslaw H. S., Jaeger J. C., *Conduction of Heat in Solids* (2nd Edn.). Clarendon Press, Oxford, 1986.

TRANSMITEREA CĂLDURII ÎNTR-O REGIUNE INFINITĂ ȘI POLINOAMELE HERMITE

(Rezumat)

Se consideră o ecuație cu derivate parțiale ce modelează transmiterea căldurii într-o regiune infinită și se obține soluția sub formă de serie ai cărei termeni sînt polinoame Hermite.

TRANSFORMATIONS DE LA CIRCULATION STANDARDÉ ADMISSIBLE DANS UN GRAPHE LATTICEL AVEC UNE DIMENSION NON ORIENTÉE

PAR

VIOREL GHIUȘ

Théorème 1. *Pour toute circulation standardé admissible [4] dans un graphe latticel avec une dimension, il existe une circulation standardé admissible dans un graphe éventail.*

Démonstration. Soit un graphe latticel avec une dimension non orienté G avec une circulation standardé admissible (f, F) (Fig. 1) où f est une orientation $\bar{b}/\bar{c}'$ ou $\bar{b}'/\bar{c}$ ou $\bar{c}/\bar{b}'$ ou $\bar{c}'/\bar{b}$ [3] de G et F une circulation dans G^f (le graphe orienté attaché). Les flèches du graphe indiquent l'orientation $f = \bar{b}/\bar{c}'$.

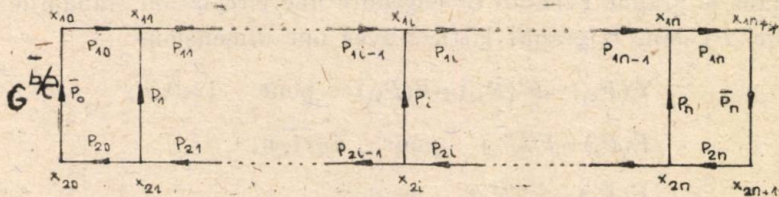


Fig. 1

Le graphe donné est transformé dans un graphe éventail en condensant les sommets x_{2i} avec $i = \overline{0, n+1}$ et les arêtes P_{2i} avec $i = \overline{0, n}$ dans le sommet O_2 , en résultant le graphe éventail $G^{\bar{b}}$ (Fig. 2).

Les fonctions de capacité c' et C' pour le graphe éventail $G^{\bar{b}}$ sont définies comme suit :

$$\begin{cases} c'(P_{1i}) = \max_k(c(P_{ki})), \\ C'(P_{1i}) = \min_k(C(P_{ki})), \end{cases} \quad k = \overline{1, 2}; \quad \begin{cases} c'(P_i) = c(P_i), \\ C'(P_i) = C(P_i), \end{cases} \quad i = \overline{1, n}; \quad \begin{cases} c'(\bar{P}_0) = c(\bar{P}_0), \\ C'(\bar{P}_n) = C(\bar{P}_n). \end{cases}$$

On définit une circulation standardé admissible $(\bar{b}, F')$ dans le graphe $G^{\bar{b}}$ comme la restriction de la fonction de circulation standardé admissible F dans le graphe donné. Pour toute arête du graphe $G^{\bar{b}}$, $F' = F$; donc les équations de conservation sont satisfaites.

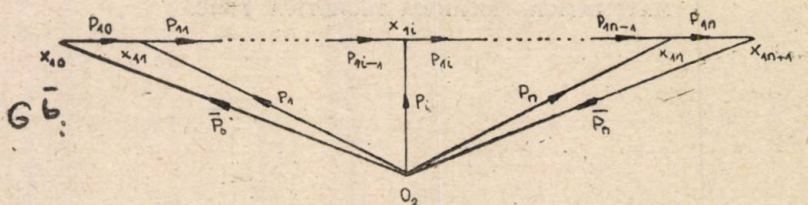


Fig. 2

Dans le sommet O_2 nous avons [4]

$$F(\bar{P}_n) = F(\bar{P}_0) + \sum (F(P_i); 1 \leq i \leq n) = F(P_{1n}).$$

Remarque 1. Le Théorème 1 peut être démontré en condensant les sommets x_{1i} avec $i = \overline{0, n+1}$ et les arêtes P_{1i} avec $i = \overline{0, n}$ dans le sommet O_1 .

Remarque 2. Réciproquement, une circulation standard admissible $(\bar{b}, F')$ dans le graphe éventail G^b (Fig. 2) peut être étendue à une circulation standard admissible $(\bar{b}/c', F)$ d'un graphe latticiel avec une dimension.

Démonstration. En scindant le sommet O_2 dans les sommets x_{2i} avec $i = \overline{0, n+1}$, lequel nous joignons avec les arêtes $P_{2i} = [x_{2i}, x_{2i+1}]$ avec $i = \overline{0, n}$. Nous définissons les fonctions de capacité sur les nouvelles arêtes: $c[P_{2i}] = 0$ et $C[P_{2i}] = \infty$ pour $i = \overline{0, n}$.

On peut facilement vérifier qu'une circulation standard admissible $(\bar{b}, F')$ dans le graphe éventail G^b engendre une circulation standard admissible $(\bar{b}/c', F)$ dans le graphe latticiel avec une dimension

$$F(P_{2i}) = F'(P_{1i}) = F(P_{1i}) \quad \text{pour} \quad i = \overline{0, n},$$

$$F(P_i) = F'(P_i) \quad \text{pour} \quad i = \overline{1, n},$$

$$F(\bar{P}_k) = F'(\bar{P}_k), \quad k = 1, 2.$$

Remarque 3. De même, s'il y a une circulation standard admissible dans le graphe éventail $G^{c'}$, nous définissons la circulation standard admissible dans le graphe latticiel avec une dimension G^b/c' , en scindant le sommet O_1 dans les sommets x_{1i} avec $i = \overline{0, n}$, lequel nous joignons avec les arêtes $P_{1i} = [x_{1i}, x_{1i+1}]$ avec $i = \overline{0, n}$.

Théorème 2. Pour tout circulation standard admissible dans un graphe latticiel avec une dimension, il existe une circulation admissible dans des graphes circuits.

Démonstration. Le graphe donné (Fig. 1) est transformé dans les sous graphes circuits $\bar{G}$ en scindant :

a) les sommets (arêtes) $x_{ki}(P_{ki})$ avec $k = 1, 2$ et $i = \overline{1, n}$ dans les sommets (arêtes) $x_{kij}(P_{kij} = [x_{kij}, x_{k(i+1)j}])$ avec $j = \overline{0, i}$;

b) les sommets x_{kn+1} avec $k = 1, 2$ dans les sommets x_{kn+1j} avec $k = 1, 2$ et $j = \overline{0, n}$;

c) l'arête $\bar{P}_n$ dans les arêtes $\bar{P}_{nj} = [x_{1n+1j}, x_{2n+1j}]$ avec $j = \overline{0, n}$.
On peut former les circuits $\bar{G}$ (Fig. 3) :

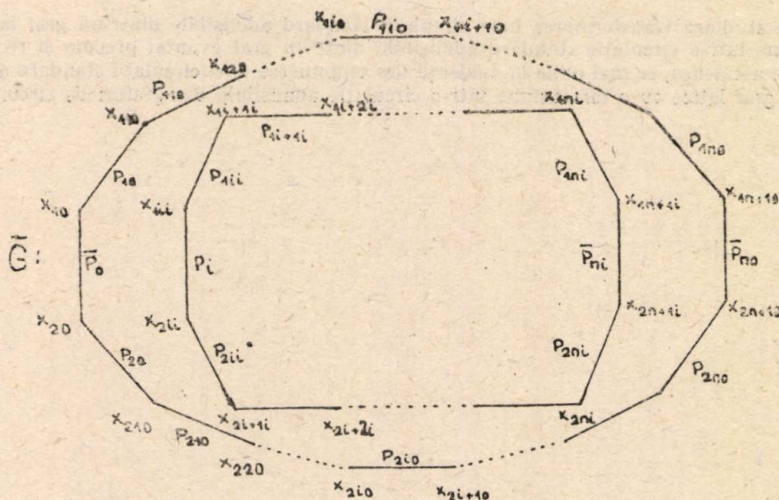


Fig. 3

$$C_0 = \{x_{10}, P_{10}, x_{110}, P_{110}, x_{120}, \dots, x_{1i0}, P_{1i0}, x_{1i+10}, \dots, x_{1n0}, P_{1n0}, \\ x_{1n+10}, \bar{P}_{n0}, x_{2n+10}, P_{2n0}, x_{2n0}, \dots, x_{2i+10}, P_{2i0}, x_{2i0}, \dots, x_{220}, P_{210}, \\ x_{210}, P_{20}, x_{20}, \bar{P}_0, x_{10}\},$$

$$C_i = \{x_{1i1}, P_{1i1}, x_{1i+1i}, P_{1i+1i}, x_{1i+2i}, \dots, x_{1ni}, P_{1ni}, x_{1n+1i}, \bar{P}_{ni}, x_{2n+1i}, \\ P_{2ni}, x_{2ni}, \dots, x_{2i+2i}, P_{2i+2i}, x_{2i+1i}, P_{2i1i}, x_{2i1i}, P_i, x_{1i1}\},$$

$$i = \overline{1, n}.$$

Nous définissons les fonctions de capacité c' et C' dans les graphes circuits $\bar{G}$ à l'aide des circulations minime et maxime 4 ; dans les graphes circuits il existe une circulation admissible [2].

Remarque 4. Réciproquement, une circulation admissible dans les graphes circuits $\bar{G}$ peut être réduite à une circulation standard admissible dans un graphe latticiel avec une dimension.

Reçue le 20 Novembre, 1991

Institut Polytechnique, Jassy,
Chaire de Mathématiques

BIBLIOGRAPHIE

1. Ford L. R., Fulkerson D. R., *Flots dans les graphes*. Gauthier-Villars, 1967.
2. Ghiuș V., *Circulație în grafuri trasate arbitrar generalizate*. Bul. Inst. Polit. Iași, **XXIII** (XXVII), 1-2, s. I, 69-73 (1977).
3. Ghiuș V., *O clasificare a circulațiilor din grafurile latici unidimensionale*. Lucrările sesiunii jubiliare Gh. Asachi, Iași, 10-12 noiembrie 1988, pp. 276-279.
4. Ghiuș V., *La circulation standard admissible dans un graphe latticiel avec une dimension* (en cours d'apparition).
5. Ore O., *Theory of Graphs*. American Society Colloq. Publ., Rhode Island, **XXXVIII** (1962).

TRANSFORMĂRI ALE UNEI CIRCULAȚII STANDARD ADMISIBILE DINTR-UN GRAF LATICE CU O DIMENSIUNE

(Rezumat)

Se studiază transformarea unei circulații standard admisibile dintr-un graf latice cu o dimensiune într-o circulație standard admisibilă dintr-un graf evantai precum și reciproc.

De asemenea, se mai pune în evidență descompunerea unei circulații standard admisibile dintr-un graf latice cu o dimensiune într-o circulație admisibilă din grafuri de circuite.

ON THE DEFORMATION OF PARABOLICAL SHELLS

BY

I. IBĂNESCU

We discuss here some deformation compatibility conditions for the deformation of a revolution parabolical shell, namely we find a condition to be satisfied between the height and the radius of basis of the shell in the case of small deformation.

Let be a revolution parabolical shell with middle surface generated by the rotating of a parabolical arc around a vertical Oz axis. The equations of the parabolical surface are

$$\begin{aligned} (1) \quad x &= \rho \cos \varphi = (az^2 + bz + c) \cos \varphi, \\ y &= \rho \sin \varphi = (az^2 + bz + c) \sin \varphi, \\ z &= z, \end{aligned}$$

where

$$(2) \quad \rho = az^2 + bz + c,$$

with $a > 0$ and

$$(3) \quad c = R, \quad b = \frac{r - R}{H} - aH.$$

Here H is the height of the shell, r and R are the radii of the basis.

Let $\mathbf{r}(z, \varphi)$ denote the position vector from a fixed origin to a generic point on the middle surface of the undeformed parabolical shell. The tangential base vector are :

$$\begin{aligned} (4) \quad \mathbf{E}_1 &= \frac{\partial \mathbf{r}}{\partial z} = \rho' \cos \varphi \mathbf{i} + \rho' \sin \varphi \mathbf{j} + \mathbf{k}, \\ \mathbf{E}_2 &= \frac{\partial \mathbf{r}}{\partial \varphi} = -\rho \sin \varphi \mathbf{i} + \rho \cos \varphi \mathbf{j}, \end{aligned}$$

where $\rho' = d\rho/dz$.

The first fundamental tensor of the undeformed middle surface is defined by the scalar product of the basis vectors :

$$\begin{aligned} (5) \quad F &= \mathbf{E}_1 \cdot \mathbf{E}_1 = 1 + \rho'^2 = A^2, \\ F &= \mathbf{E}_1 \cdot \mathbf{E}_2 = 0, \\ G &= \mathbf{E}_2 \cdot \mathbf{E}_2 = \rho^2 = B^2. \end{aligned}$$

Because $F=0$, it follows that the coordinates lines are orthogonales.

The unit normal vector $\mathbf{n}$ to the undeformed middle surface is defined by

$$(6) \quad \mathbf{n} = \frac{\mathbf{E}_1 \times \mathbf{E}_2}{|\mathbf{E}_1 \times \mathbf{E}_2|} = \frac{1}{A} (\cos \varphi \mathbf{i} + \sin \varphi \mathbf{j} - \rho' \mathbf{k}).$$

The second fundamental tensor of the undeformed middle surface is defined by

$$(7) \quad \begin{aligned} L &= \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial z^2} = \frac{\rho''}{A}, \\ M &= \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial z \partial \varphi} = 0, \\ N &= \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial \varphi^2} = -\frac{\rho}{A}. \end{aligned}$$

The principal curvatures of the undeformed middle surface are

$$(8) \quad k_1 = -\frac{L}{A^2} = -\frac{\rho''}{A^3}, \quad k_2 = -\frac{N}{B^2} = \frac{1}{AB}.$$

The equations of Mainardi - Codazzi - Gauss are

$$(9) \quad (k_2 B)_{,z} = k_1 B_{,z}, \quad \left(\frac{1}{A} B_{,z} \right)_{,z} + A B k_1 k_2 = 0.$$

At each point $M(z, \varphi)$ of the undeformed middle surface the local basis is defined as

$$(10) \quad \begin{aligned} \mathbf{e}_1 &= \frac{1}{A} \mathbf{E}_1 = \frac{1}{A} \mathbf{r}_{,z}, \\ \mathbf{e}_2 &= \frac{1}{B} \mathbf{E}_2 = \frac{1}{B} \mathbf{r}_{,\varphi}, \\ \mathbf{e}_n &= \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n}. \end{aligned}$$

A point outside the middle surface may be identified by the coordinates z, φ of its projection on the middle surface and its distance to the middle surface $\gamma \in [-h, h]$ as its third coordinate, where h is the shell thickness.

The position vector $\mathbf{R}$ of an arbitrary point of the shell can be written

$$(11) \quad \mathbf{R}(z, \varphi, \gamma) = \mathbf{r}(z, \varphi) + \gamma \mathbf{n}.$$

From (10) and (11) we have

$$(12) \quad \mathbf{R}_{,z} = A(1 + k_1 \gamma) \mathbf{e}_1, \quad \mathbf{R}_{,\varphi} = B(1 + k_2 \gamma) \mathbf{e}_2, \quad \mathbf{R}_{,\gamma} = \mathbf{n}.$$

Let (U, V, W) denote the components of the displacement vector of an arbitrary point of the shell.

For the components of the strain tensor we have :

$$\begin{aligned}
 e_{zz} &= \frac{1}{A(1+k_1\gamma)} \frac{\partial U}{\partial z} + \frac{k_1}{1+k_1\gamma} W, \\
 e_{z\varphi} &= \frac{A}{B(1+k_2\gamma)} \frac{\partial U}{\partial \varphi} + \frac{B}{A} \frac{i+k_2\gamma}{1+k_1\gamma} \frac{\partial}{\partial z} \left(\frac{V}{B(1+k_2\gamma)} \right), \\
 e_{\gamma\gamma} &= \frac{\partial W}{\partial \gamma}, \\
 e_{\varphi\varphi} &= \frac{1}{B(1+k_2\gamma)} \frac{\partial V}{\partial \varphi} + \frac{\rho'}{AB(1+k_2\gamma)} U + \frac{k_2}{1+k_2\gamma} W, \\
 e_{z\gamma} &= A(1+k_1\gamma) \frac{\partial}{\partial \gamma} \left(\frac{U}{A(1+k_1\gamma)} \right) + \frac{1}{A(1+k_1\gamma)} \frac{\partial W}{\partial z}, \\
 e_{\varphi\gamma} &= B(1+k_2\gamma) \frac{\partial}{\partial \gamma} \left(\frac{V}{B(1+k_2\gamma)} \right) + \frac{1}{B(1+k_2\gamma)} \frac{\partial W}{\partial \varphi}.
 \end{aligned}
 \tag{13}$$

Kirchhoff-Love Hypotheses. The basic assumption of the classical Kirchhoff-Love theory of thin shells is that the normals to the undeformed middle surface move to the normals of the deformed middle surface without any change in length. According to this hypotheses we can write

$$e_{z\gamma} = e_{\varphi\gamma} = e_{\gamma\gamma} = 0 \tag{14}$$

and from (13) it follows

$$\begin{aligned}
 W &= w(z, \varphi), \\
 U &= u(z, \varphi) - O_1, \\
 V &= v(z, \varphi) - O_2,
 \end{aligned}
 \tag{15}$$

where

$$u(z, \varphi) = U(z, \varphi, 0), \quad v(z, \varphi) = V(z, \varphi, 0) \tag{16}$$

are the components of displacement vector for a generic point of the middle surface and

$$O_1 = \frac{1}{A} \frac{\partial w}{\partial z} - k_1 u, \quad O_2 = \frac{1}{B} \frac{\partial w}{\partial \varphi} - k_2 v \tag{17}$$

are the components of the rotation around the normal.

Introducing (15) into (13), we find

$$\begin{aligned}
 e_{zz} &= \frac{1}{1+k_1\gamma} (\varepsilon_1 + \kappa_1\gamma), \\
 e_{\varphi\varphi} &= \frac{1}{1+k_2\gamma} (\varepsilon_2 + \kappa_2\gamma), \\
 e_{z\varphi} &= \frac{1}{1+k_1\gamma} (\omega_1 + \tau_1\gamma) + \frac{1}{1+k_2\gamma} (\omega_2 + \tau_2\gamma),
 \end{aligned}
 \tag{18}$$

where

$$\begin{aligned}
 \varepsilon_1 &= \frac{1}{A} \frac{\partial u}{\partial z} + k_1 w, & \varepsilon_2 &= \frac{1}{B} \frac{\partial v}{\partial \varphi} + \frac{1}{AB} \rho' u + k_2 w, \\
 \omega_1 &= \frac{1}{A} \frac{\partial v}{\partial z}, & \omega_2 &= \frac{1}{B} \frac{\partial u}{\partial \varphi} - \frac{\rho'}{AB} v, \\
 \kappa_1 &= -\frac{1}{A} \frac{\partial O_1}{\partial z}, & \kappa_2 &= \frac{1}{B} \frac{\partial u}{\partial \varphi} - \frac{\rho'}{AB} O_1, \\
 \tau_1 &= -\frac{1}{A} \frac{\partial O_2}{\partial z}, & \tau_2 &= -\frac{1}{B} \frac{\partial O_1}{\partial \varphi} + \frac{1}{AB} \rho' O_2.
 \end{aligned}
 \tag{19}$$

We observe that

$$\tau_1 + k_1 \omega_2 = \tau_2 + k_2 \omega_1.
 \tag{20}$$

Denoting

$$\begin{aligned}
 \varepsilon_{12} &= \omega_1 + \omega_2, \\
 \kappa_{12} &= \tau_1 + k_1 \omega_2 = \tau_2 + k_2 \omega_1,
 \end{aligned}
 \tag{21}$$

from (19) we have

$$\varepsilon_{12} = \frac{1}{B} \frac{\partial u}{\partial \varphi} + \frac{B}{A} \frac{\partial}{\partial z} \left(\frac{u}{B} \right),
 \tag{22}$$

$$\kappa_{12} = -\frac{1}{AB} \left(\frac{\partial^2 w}{\partial z \partial \varphi} - \frac{1}{B} \rho' \frac{\partial w}{\partial \varphi} \right) + k_1 \frac{1}{B} \frac{\partial n}{\partial \varphi} + k_2 \frac{B}{A} \frac{\partial}{\partial z} \left(\frac{v}{B} \right).
 \tag{23}$$

The set of variables $\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \kappa_1, \kappa_2, \kappa_{12}$ defines the deformation of the middle surface.

Compatibility Conditions. The fundamental tensors $E=A^2, G, F=B^2, L, M, N$ of the middle surface of a undeformed shell satisfy the equations of Gauss and Mainardi-Codazzi

$$\begin{aligned}
 \frac{\partial L}{\partial \varphi} - \frac{\partial M}{\partial z} - \Gamma_{12}^1 L + (\Gamma_{11}^1 - \Gamma_{12}^2) M + \Gamma_{11}^2 N &= 0, \\
 \frac{\partial N}{\partial z} - \frac{\partial M}{\partial \varphi} + \Gamma_{22}^1 L + (\Gamma_{22}^2 - \Gamma_{12}^1) M - \Gamma_{12}^2 N &= 0, \\
 \frac{LN - M^2}{A^2 B^2} - \frac{1}{AB} \frac{\partial}{\partial z} \left(\frac{\rho''}{A} \right) &= 0,
 \end{aligned}
 \tag{24}$$

where Γ_{ij}^k are the Christoffel symbols.

Let $E'=A'^2, F', G'=B'^2, L', M', N'$ denote the fundamental tensors of the middle surface of the deformed shell. For this we write the Gauss-Mainardi-Codazzi equations:

$$\begin{aligned}
 \frac{\partial}{\partial z} (B \kappa_2) - \rho' \kappa_1 - A \frac{\partial}{\partial \varphi} \kappa_{12} + k_1 \left(-\frac{\partial}{\partial z} (B \varepsilon_2) + \rho' \varepsilon_1 + A \frac{\partial \varepsilon_{12}}{\partial \varphi} \right) &= 0, \\
 A \frac{\partial \kappa_1}{\partial \varphi} - \frac{\partial}{\partial z} (B \kappa_{12}) - \rho' \kappa_{12} + k_2 \left(-A \frac{\partial \varepsilon_1}{\partial \varphi} + \frac{\partial}{\partial z} (B \varepsilon_{12}) + \frac{k_1}{k_2} \rho' \varepsilon_{12} \right) &= 0,
 \end{aligned}$$

$$(25) \quad k_2 \chi_1 + k_1 \chi_2 - \frac{1}{AB} \left[\frac{\partial}{\partial z} \frac{1}{A} \left(-\frac{\partial}{\partial z} (B \varepsilon_2) + \rho' \varepsilon_1 + \frac{A}{2} \frac{\partial \varepsilon_{12}}{\partial \varphi} \right) + \right. \\ \left. + \frac{1}{B} \left(-A \frac{\partial^2 \varepsilon_1}{\partial \phi^2} + \frac{1}{2} \frac{\partial^2}{\partial z \partial \varphi} (B \varepsilon_{12}) + \frac{1}{2} \rho' \varepsilon_{12} \right) \right] = 0.$$

In what follows, we consider the deflection $w(z, \varphi)$ and the components of rotation around the normal $O_1(z, \varphi)$, $O_2(z, \varphi)$ as functions which characterise the deformation of the middle surface.

Using (17), (22), (23) we can write :

$$(26) \quad \begin{aligned} \varepsilon_1 &= -\frac{2\rho'}{A} w_{,z} + \frac{1}{A^2 k_1} w_{,zz} + 3\rho' O_1 - \frac{1}{A k_1} O_{1,z} + k_1 w, \\ \varepsilon_2 &= \frac{A}{B} w_{,\varphi\varphi} - A O_{2,\varphi} + \frac{\rho' k_2}{A k_1} w_{,z} - \frac{\rho' k_2}{k_1} O_1 + k_2 w, \\ \varepsilon_{12} &= \frac{k_2 + k_1}{k_1} w_{,z\varphi} - \frac{1}{B k_1} O_{1,\varphi} - \rho' A (k_1 + k_2) w_{,\varphi} + \rho' \frac{k_1}{k_2} O_2 - B O_{2,z}, \\ \chi_1 &= -\frac{1}{A} O_{1,z}, \\ \chi_2 &= -\frac{1}{B} O_{2,\varphi} - k_2 \rho' O_1, \\ \chi_{12} &= k_2 w_{,z\varphi} - k_1 k_2 A \rho' w_{,\varphi} - \frac{1}{B} O_{1,\varphi} - \frac{1}{A} O_{2,z} + \rho' k_1 O_2. \end{aligned}$$

Substituting (26) into (25) we have

$$(27) \quad -\frac{2\rho'^2}{A} (2k_1 + k_2) w_{,z} = 0,$$

$$(28) \quad \begin{aligned} A k_1^2 (1 + 4\rho'^2) O_{2,\varphi} + 2\rho' A k_2 (k_1 - k_2) w_{,z} + \frac{\rho'^2}{A^2} \frac{k_2}{k_1} (k_1 - k_2) w_{,zz} - \\ - \rho' A k_2 (k_1 + k_2) w_{,z\varphi} = 0. \end{aligned}$$

The second compatibility condition is identically verified.

Because the deflection w is a function of z , the first compatibility condition (42) is reduced to

$$(29) \quad 2k_1 + k_2 = 0.$$

By means of (2), (5), (8) and (29) we may write

$$(30) \quad b^2 - 4ac = -1 < 0.$$

The geometric interpretation of this relation is that the parabolical arc (2) have not the common points with Oz axis.

Introducing (3) into (30) we find the equation which determines the coefficient a , namely

$$(31) \quad a^2 H^2 - 2a(R+r) + \frac{(R-r)^2}{H^2} + 1 = 0.$$

Because a is positive it results

$$(32) \quad 4Rr > H^2,$$

and it follows that the height and the basis radius of the shell must verify this condition.

Received 27 October, 1990

Polytechnic Institute, Jassy,
Department of Theoretical
Mechanics

REFERENCES

1. Подстригач И. С., *Термоупругость тонких оболочек*. „Наукова думка“, Киев, 1978.
2. Visarion V., *Elemente pentru calculul plăcilor curbe subțiri*. Edit. Acad. R.P.R., Buc., 1961.

ASUPRA DEFORMĂRII PLĂCILOR PARABOLICE

(Rezumat)

Se analizează condițiile de compatibilitate a deformațiilor unei plăci parabolice de revoluție, stabilindu-se o condiție pe care trebuie să o verifice înălțimea și raza bazei plăcii în cazul deformațiilor mici.

RECIPROCITY THEOREMS FOR THE DYNAMIC PROBLEM OF COUPLED GENERALIZED THERMOELASTICITY WITH MICROSTRUCTURE AND SOME APPLICATIONS

BY

D. VIERU and SILVIA CIUMAȘU

0. Introduction. In [7] we derived a generalized dynamical theory of thermoelasticity with microstructure and in [8] we proved a uniqueness theorem, an alternative formulations of the problem and a variational theorem in linear dynamic theory.

It is the purpose of this paper to present reciprocity theorems and applications in the dynamic problem of coupled generalized thermoelasticity with microstructure. The reciprocity theorems are derived without using Laplace transform, by extending results from [3]...[6].

1. Basic Equations. Let V be a regular domain in the three-dimensional space occupied by an elastic body with microstructure whose boundary is ∂V . We refer the motion of continuum to a fixed system of rectangular cartesian axes Ox_i ($i=1, 2, 3$).

The basic equations in generalized theory of linear thermoelasticity of a homogeneous with microstructure and centrosymmetric continuum are :

i) the *motion equations*

$$(1.1) \quad \begin{aligned} \sigma_{ij,j} - \tau_{ij,j} + f_i &= \rho \ddot{u}_i, \\ -\mu_{ij} - \tau_{ij} + f_{ij} &= \rho I_{ik} \ddot{\phi}_{kj}; \end{aligned}$$

ii) the *energy equation*

$$(1.2) \quad \rho \theta_0 \dot{S} = \rho r - q_{i,i};$$

iii) the *constitutive equations*

$$(1.3) \quad \begin{aligned} \sigma_{ij} &= A_{ij}(\theta + \alpha \dot{\theta}) + A_{ijkl} \varepsilon_{kl} + B_{ijkl} e_{kl} + E_{kl} r_{kl}, \\ \tau_{ij} &= C_{ij}(\theta + \alpha \dot{\theta}) + E_{ijkl} \varepsilon_{kl} + F_{ijkl} e_{kl} + C_{ijkl} r_{kl}, \\ \mu_{ij} &= B_{ij}(\theta + \alpha \dot{\theta}) + D_{kl} \varepsilon_{kl} + B_{ijkl} e_{kl} + F_{kl} r_{kl}, \\ t_{ij} &= \sigma_{ij} + \tau_{ij} = \sigma_{ij} - \tau_{ji}, \quad m_{ij} = \mu_{ij} + \tau_{ij}, \\ q_i &= -\theta_0 k_{ij} \theta_{,j}, \\ \rho S &= a + d\theta + h\dot{\theta} - A_{ij} \varepsilon_{ij} - B_{ij} e_{ij} - C_{ij} r_{ij}; \end{aligned}$$

iv) the kinematic relations

$$(1.4) \quad \begin{aligned} 2\varepsilon_{ij} &= u_{i,j} + \mu_{j,i}, & 2e_{ij} &= \varphi_{ij} + \varphi_{ji}, \\ 2r_{ij} &= (\varphi_{ij} - \varphi_{ji}) - (u_{i,j} - u_{j,i}). \end{aligned}$$

In the above equations we use the following notations: u_i , φ_{ij} , θ , θ_0 , t_{ij} , m_{ij} , f_i , f_{ij} , r , q_i , S , ρ respectively — the displacement, dipolar displacement, temperature difference, reference temperature, stress, couple stress, body force, dipolar force, heat supply, heat flux vector, specific entropy, density. The thermoelastic coefficients are subject to the symmetry

$$(1.5) \quad \begin{aligned} A_{ij} &= A_{ji}, & B_{ij} &= B_{ji}, & C_{ij} &= -C_{ji}, & I_{ij} &= I_{ji}, & k_{ij} &= k_{ji}, \\ A_{ijkl} &= A_{klij} = A_{jikl}, & B_{ijkl} &= B_{jikl} = B_{klij}, \\ C_{ijkl} &= -C_{jikl} = -C_{ijlk}, & D_{ijkl} &= D_{jikl} = D_{ijlk}, \\ E_{ijkl} &= -E_{jikl} = E_{ijlk}, & F_{ijkl} &= F_{jikl} = -F_{ijlk}. \end{aligned}$$

To Eqs. (1.1)–(1.4) we adjoin the boundary conditions

$$(1.6) \quad \begin{aligned} u_i &= \bar{u}_i \text{ on } \bar{\Sigma}_1 \times [0, t_0], & t_{ij} n_i &= \bar{t}_j \text{ on } \Sigma_2 \times [0, t_0], \\ \theta &= \bar{\theta} \text{ on } \bar{\Sigma}_3 \times [0, t_0], & q_i n_i &= \bar{q} \text{ on } \Sigma_4 \times [0, t_0], \end{aligned}$$

where Σ_k ($k=1, 2, 3, 4$) denote subsets of ∂V so that: $\bar{\Sigma}_1 \cup \Sigma_2 = \bar{\Sigma}_3 \cup \Sigma_4 = \partial V$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Phi$, n_j is the unit outward normal to ∂V ;

v) the initial conditions

$$(1.7) \quad (u_i, \dot{u}_i, \varphi_{ij}, \dot{\varphi}_{ij}, \theta, \dot{\theta})(x, 0) = (u_i^0, v_i^0, \varphi_{ij}^0, V_{ij}^0, \theta_0, \Theta_0)(x), \quad x \in \bar{V}.$$

In these conditions $\tilde{u}_i$, $\tilde{t}_j$, $\tilde{\theta}$, $\tilde{q}$, u_i^0 , v_i^0 , φ_{ij}^0 , v_{ij}^0 , θ_0 , Θ_0 are prescribed functions.

In [8] we presented an alternative formulation of the linear mixed problem.

An admissible state $\{u_i, \varphi_{ij}, \varepsilon_{ij}, e_{ij}, r_{ij}, t_{ij}, m_{ij}, \theta, S, q_i\}$ is a solution of the mixed problem if and only if it meets the field Eq. (1.3), (1.4), (1.6) and

$$(1.8) \quad \begin{aligned} I_i &= g * t_{ji,i} + F_i = \rho u_i, \\ M_{ij} &= g * (-m_{ij}) + F_{ij} = \rho I_{ik} \varphi_{kj}, \\ \rho S &= -\frac{1}{\theta_0} l * q_{i,i} + w, \text{ on } V * [0, t_0], \end{aligned}$$

where

$$(1.9) \quad \begin{aligned} g(t) &= l, & l(t) &= l, & F_i &= g * f_i + \rho(t v_i^0 + u_i^0), \\ F_{ij} &= g * f_{ij} + \rho I_{ik}(t V_{kj}^0 + \varphi_{kj}^0), & w &= \frac{\rho}{\theta_0} l * r + \rho S_0. \end{aligned}$$

2. Reciprocity Theorems. We consider the body subjected at two different systems of elastic loadings

$$L^{(\beta)} = \{f_i^{(\beta)}, f_{ij}^{(\beta)}, r^{(\beta)}, \tilde{u}_i^{(\beta)}, \tilde{l}_j^{(\beta)}, \tilde{\theta}^{(\beta)}, \tilde{q}^{(\beta)}, u_i^{0(\beta)}, V_i^{0(\beta)}, \varphi_{ij}^{0(\beta)}, \\ V_{ij}^{0(\beta)}, \theta^{0(\beta)}, \Theta_0^{(\beta)}\} \quad \text{with} \quad \dot{\theta}^{(\beta)} = 0, \dot{\Theta}_0^{(\beta)} = 0,$$

and the two corresponding states

$$\varphi^{(\beta)} = \{u_i^{(\beta)}, \varphi_{ij}^{(\beta)}, \theta^{(\beta)}\} \quad \beta = 1, 2.$$

Theorem 2.1. *If a thermoelastic solid is subjected at two different systems of loadings $L^{(\beta)}$ ($\beta=1, 2$), then between the corresponding configurations $\varphi^{(\beta)}$ takes place the following reciprocity relation*

$$(2.1) \quad \int_V [F_i^{(1)} * u_i^{(2)} + F_{ij}^{(1)} * \varphi_{ij}^{(2)} - (g + \alpha l) * w^{(1)} * \theta^{(2)} + a(g + \alpha l) * \theta^{(2)}] dV + \\ + \int_{\partial V} g * \left[\tilde{l}_i^{(1)} * u_i^{(2)} + \frac{1}{\theta_0} l * \tilde{q}^{(1)} * \theta^{(2)} + \frac{\alpha}{\theta_0} \tilde{q}^{(1)} * \theta^{(2)} \right] dA = \\ = \int_V [F_i^{(2)} * u_i^{(1)} + F_{ij}^{(2)} * \varphi_{ij}^{(1)} - (g + \alpha l) * w^{(2)} * \theta^{(1)} + a(g + \alpha l) * \theta^{(1)}] dV + \\ + \int_{\partial V} g * \left[\tilde{l}_i^{(2)} * u_i^{(1)} + \frac{1}{\theta_0} l * \tilde{q}^{(2)} * \theta^{(1)} + \frac{\alpha}{\theta_0} \tilde{q}^{(2)} * \theta^{(1)} \right] dA.$$

Proof. We will use the method given in [3]. On the basis of relations (1.3), (1.5) and using the convolution properties, we derive

$$(2.2) \quad A_{\beta\gamma} = A_{\gamma\beta} \quad (\gamma, \beta = 1, 2),$$

where

$$(2.3) \quad A_{\beta\gamma} = \sigma_{ij}^{(\beta)} * \varepsilon_{ij}^{(\gamma)} + \mu_{ij}^{(\beta)} * e_{ij}^{(\gamma)} + \tau_{ij}^{(\beta)} * r_{ij}^{(\gamma)} - [\rho S^{(\beta)} - (a + h\dot{\theta}^{(\beta)})] * \theta^{(\gamma)} - \\ - \alpha [\rho S^{(\beta)} - (a + d\dot{\theta}^{(\beta)})] * \dot{\theta}^{(\gamma)}.$$

If we introduce the notation

$$(2.4) \quad U_{\beta\gamma} = \int_V (g * A_{\beta\gamma}) dV,$$

from (2.2) we have

$$(2.5) \quad U_{12} = U_{21}.$$

Using the Eqs. (1.4), (1.8) and the divergence theorem we get

$$U_{\beta\gamma} = \int_V g * \left[\tilde{l}_i^{(\beta)} * u_i^{(\gamma)} + \frac{1}{\theta_0} l * \tilde{q}^{(\beta)} * \theta^{(\gamma)} + \frac{\alpha}{\theta_0} \tilde{q}^{(\beta)} * \theta^{(\gamma)} \right] dA +$$

$$(2.6) \quad + \int_V [(F_i^{(\beta)} - \rho u_i^{(\beta)}) * u_i^{(\gamma)} + (F_{ij}^{(\beta)} - \rho I_{ik} \varphi_{kj}^{(\beta)}) * \varphi_{ij}^{(\gamma)} - (g + \alpha l) * w^{(\beta)} * \theta^{(\gamma)} + \\ + \alpha(g + \alpha l) * \theta^{(\gamma)} + (h + \alpha d)l * \theta^{(\beta)} * \theta^{(\gamma)} + k_{ij}l * (g + \alpha l) * \theta_i^{(\beta)} * \theta_j^{(\gamma)}] dV.$$

From (2.5) and (2.6) we obtain (2.1).

In zero initial conditions we have

$$(2.7) \quad F_i^{(\beta)} = g * f_i^{(\beta)}, \quad F_{ij}^{(\beta)} = g * f_{ij}^{(\beta)}, \quad w^{(\beta)} = \frac{\rho}{\theta_0} l * r^{(\beta)} + c.$$

From (2.1) and (2.7) we obtain

$$(2.8) \quad g * \left\{ \int_V \left[f_i^{(1)} * u_i^{(2)} + f_{ij}^{(1)} * f_{ij}^{(2)} - \frac{\rho}{\theta_0} l * r^{(1)} * (\theta^{(2)} + \alpha \dot{\theta}^{(2)}) \right] dV + \right. \\ \left. + \int_{\partial V} \left[\tilde{t}_i^{(1)} * u_i^{(2)} + \frac{1}{\theta_0} l * q^{(1)} * \theta^{(2)} + \frac{\alpha}{\theta_0} \tilde{q}^{(1)} * \theta^{(2)} \right] dA - \right. \\ \left. - \int_V \left[f_i^{(2)} * u_i^{(1)} + f_{ij}^{(2)} * \varphi_{ij}^{(1)} - \frac{\rho}{\theta_0} l * r^{(2)} * (\theta^{(1)} + \alpha \dot{\theta}^{(1)}) \right] dV - \right. \\ \left. - \int_{\partial V} \left[\tilde{t}_i^{(2)} * u_i^{(1)} + \frac{1}{\theta_0} l * q^{(2)} * \theta^{(1)} + \frac{\alpha}{\theta_0} \tilde{q}^{(2)} * \theta^{(1)} \right] dA \right\} = 0.$$

Because $g=0$ from (2.8) we have

$$(2.9) \quad \int_V \left[f_i^{(1)} * u_i^{(2)} + f_{ij}^{(1)} * \varphi_{ij}^{(2)} - \frac{\rho}{\theta_0} l * r^{(1)} * (\theta^{(2)} + \alpha \dot{\theta}^{(2)}) \right] dV + \\ + \int_{\partial V} \left[\tilde{t}_i^{(1)} * u_i^{(2)} + \frac{1}{\theta_0} l * \tilde{q}^{(1)} * (\theta^{(2)} + \alpha \dot{\theta}^{(2)}) \right] dA = \\ = \int_V \left[f_i^{(2)} * u_i^{(1)} + f_{ij}^{(2)} * \varphi_{ij}^{(1)} - \frac{\rho}{\theta_0} l * r^{(2)} * (\theta^{(1)} + \alpha \dot{\theta}^{(1)}) \right] dV + \\ + \int_{\partial V} \left[\tilde{t}_i^{(2)} * u_i^{(1)} + \frac{1}{\theta_0} l * \tilde{q}^{(2)} * (\theta^{(1)} + \alpha \dot{\theta}^{(1)}) \right] dA.$$

Using the relations

$$(2.10) \quad u_i^{(\beta)} = l * \dot{u}_i^{(\beta)}, \quad \varphi_{ij}^{(\beta)} = l * \dot{\varphi}_{ij}^{(\beta)}, \quad l \neq 0,$$

from (2.9) we obtain

$$\begin{aligned}
 & \int_V \left[f_i^{(1)} * \dot{u}_i^{(2)} + f_{ij}^{(1)} * \dot{\varphi}_{ij}^{(2)} - \frac{\rho}{\theta_0} r^{(1)} * (\theta^{(2)} + \alpha \dot{\theta}^{(2)}) \right] dV + \\
 & + \int_{\partial V} \left[\tilde{t}_i^{(1)} * \dot{u}_i^{(2)} + \frac{1}{\theta_0} \tilde{q}^{(1)} * (\theta^{(2)} + \alpha \dot{\theta}^{(2)}) \right] dA = \\
 (2.11) \quad & = \int_V \left[f_i^{(2)} * \dot{u}_i^{(1)} + f_{ij}^{(2)} * \dot{\varphi}_{ij}^{(1)} - \frac{\rho}{\theta_0} r^{(2)} * (\theta^{(1)} + \alpha \dot{\theta}^{(1)}) \right] dV + \\
 & + \int_{\partial V} \left[\tilde{t}_i^{(2)} * \dot{u}_i^{(1)} + \frac{1}{\theta_0} \tilde{q}^{(2)} * (\theta^{(1)} + \alpha \dot{\theta}^{(1)}) \right] dA.
 \end{aligned}$$

3. Some Applications of the Reciprocity Theorems. We introduce the notations

$$(3.1) \quad u_i = h_i, \quad \varphi_{ij} = h_{3i+j} = h_m, \quad h_{i3} = -\frac{\rho}{\theta_0} (l * \theta + \alpha \dot{\theta}),$$

$$f_i = R_i, \quad f_{ij} = R_{3i+j} = R_m, \quad r = R_{13}, \quad i, j = 1, 2, 3; \quad m = 4, \dots, 12.$$

In this section we consider the linear mixed problem (1.1)–(1.4) in zero initial conditions and $\Sigma_1 = \Sigma_3 = \Phi$.

We assume that the body is subjected to the following loading systems

$$(3.2) \quad R_p^{(n)} = \delta(x - \xi) \delta(t) \delta_{pn}, \quad \tilde{t}_i = 0, \quad \tilde{q} = 0, \quad i = 1, 2, 3; \quad n, p = 1, 2, \dots, 13,$$

where δ is the Dirac function.

This loading systems will generate respectively displacements and temperature fields compatible with the thermoelasticity equations, denoted by

$$(3.3) \quad H_p^{(n)}(x, \xi, t), \quad \Theta^{(n)}(x, \xi, t), \quad n = 1, 2, \dots, 13; \quad p = 1, 2, \dots, 12.$$

The functions (3.3) will be called „fundamental solutions“ relative to the domain V .

Using the reciprocity relations (2.9), (2.11) for the loading systems $\{f_i, f_{ij}, r, \tilde{t}_i, \tilde{q}\}$ which will generate fields $\{u_i, \varphi_{ij}, \theta\}$, and for the loading systems (3.2), we obtain

$$\begin{aligned}
 h_n(\xi, t) = & \int_V \left[[R_p * H_p^{(n)} - \frac{\rho}{\theta_0} R_{13} * (l * \Theta^{(n)} + \alpha \dot{\Theta}^{(n)})] dV + \right. \\
 (3.4) \quad & \left. + \int_{\partial V} \left[\tilde{t}_i * H_i^{(n)} + \frac{1}{\theta_0} \tilde{q} * (l * \Theta^{(n)} + \alpha \dot{\Theta}^{(n)}) \right] dA, \right.
 \end{aligned}$$

where $i = 1, 2, 3; p = 1, 2, \dots, 12; n = 1, 2, \dots, 13$.

These applications of the reciprocity theorems generalize the results from [5], [3], [6].

Received November 15, 1992

Polytechnic Institute, Jassy,
Department of Theoretical Mechanics

REFERENCES

1. Green A. F., Lindsay K. A., J. of Elasticity, **2**, 1, 1—7 (1972).
2. Soos E., *Modele discrete și continue ale solidelor*. Ed. științifică, Buc., 1974.
3. Ieșan D., Memorie dell Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat., Ser. 4^a, 17 (1974).
4. Ieșan D., Rev. Roum. Math. Pures et Appl., **XV**, 8, 1181—1195 (1970).
5. Ionescu-Cazimir V., Bull. Acad. Pol. Sci., Ser. Sci. Tech., **XII**, 9, 481, 667 (1964).
6. Rusu El., Bul. Inst. Polit. Iași, **XXIV(XXVIII)**, 3—4, s. IV (1978).
7. Vieru D., Ciumașu S. G., Bul. Inst. Polit. Iași, **XXXVII(XLI)**, 1—4, s. I (1991).
8. Ciumașu S. G., Vieru D., Bul. Inst. Polit. Iași (to appear).

TEOREME DE RECIPROCITATE PENTRU PROBLEMA DINAMICĂ A
TERMOELASTICITĂȚII CUPLATE GENERALIZATE CU
MICROSTRUCTURĂ ȘI UNELE APLICAȚII

(Rezumat)

Se prezintă teoreme de reciprocitate și unele aplicații ale acestora, constind în exprimarea soluției problemei cu ajutorul funcțiilor Green corespunzătoare domeniului ocupat de corp.

VARIATIONAL THEOREMS IN LINEAR THEORY OF MICROPOLAR VISCOELASTICITY

BY

D. MANOLE

0. Introduction. Variational theorems in the linear theory of viscoelasticity are given in [6], [7], [10]. In the case of linear micropolar viscoelasticity theory a variational theorems is given in [2]. In the theory of linear micropolar elasticity we find such theorem in [4]. Variational theorems for elastic media with microstructure are given in [1], [8], [11].

Variational theorems in the linear theory of viscoelasticity with microstructure are given in [9], [12].

In the present paper some variational theorems in linear micropolar viscoelasticity theory will be presented. The first two theorems are deduced by taking into account the reciprocity theorem given in [2], in a convenient form, following the method from [3] in the case of linear viscoelasticity. The third theorem is analogous to the thermoelasticity theorem [5], and the last one is for the dynamic theory of linear viscoelasticity given in [7].

1. Basic Equations. Consider a viscoelastic medium occupying the Ω domain of the three-dimensional Euclidian space with closure $\bar{\Omega}$ and (regular) boundary Σ . The basic equations of linear micropolar viscoelasticity theory are:

i) *Equations of motion*

$$(1.1) \quad \begin{aligned} t_{ji,j} + F_i &= \rho u_i^{(2)}, \\ m_{ji,j} + \varepsilon_{ijk} t_{jk} + M_i &= I_{ij} \varphi_j^{(2)}; \end{aligned}$$

ii) *Geometrical equations*

$$(1.2) \quad \begin{aligned} \gamma_{ij} &= u_{j,i} + \varepsilon_{jlm} \varphi_m, \\ \kappa_{ij} &= \varphi_{j,i}; \end{aligned}$$

iii) *Constitutive equations*

$$(1.3) \quad \begin{aligned} t_{ij} &= \int_{-\infty}^t A_{ijkl}(x, t-\tau) \gamma_{kl}^{(1)}(x, \tau) d\tau + \int_{-\infty}^t B_{ijkl}(x, t-\tau) \kappa_{kl}^{(1)}(x, \tau) d\tau, \\ m_{ij} &= \int_{-\infty}^t B_{klij}(x, t-\tau) \gamma_{kl}^{(1)}(x, \tau) d\tau + \int_{-\infty}^t C_{ijkl}(x, t-\tau) \kappa_{kl}^{(1)}(x, \tau) d\tau, \end{aligned}$$

where

$$A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij}.$$

The viscoelastic medium remembers its deformation state possessed until t_0 moment by means of constitutive relations.

Without limiting the generality, we can take $t_0 = 0$.

Consider $S^0 = \{u_i^0, \varphi_i^0, \gamma_{ij}^0, \kappa_{ij}^0, t_{ij}^0, m_{ij}^0\}$ the initial history and denote with S the assembly of functions $u_i, \varphi_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}$ defined on $\Omega \times (-\infty, 0)$. The conditions of the initial history is

$$(1.4) \quad S = S^0.$$

Obviously, from (1.4) we have

$$(1.5) \quad u_i(x, 0) = a_i(x), \quad u_i^{(1)}(x, 0) = b_i(x), \quad \varphi_i(x, 0) = c_i(x), \quad \varphi_i^{(1)}(x, 0) = d_i(x).$$

In case of a mixed problem, the boundary conditions are

$$(1.6) \quad \begin{aligned} u_i &= \tilde{u}_i \text{ on } \Sigma_1 \times (0, \infty), & \varphi_i &= \tilde{\varphi}_i \text{ on } \Sigma_3 \times (0, \infty), \\ t_i &= t_{ij} n_j = \tilde{t}_i \text{ on } \Sigma_2 \times (0, \infty), & m_i &= m_{ij} n_j = \tilde{m}_i \text{ on } \Sigma_4 \times (0, \infty), \end{aligned}$$

where $\Sigma_r (r=1, 2, 3, 4)$ are parts of Σ so that $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \Sigma$, $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$. In these relations the following notations have been used: t_{ij} — components of the stress tensor, m_{ij} — components of the couple stress tensor, u_i — components of the displacement vector, φ_i — components of the microrotation vector, γ_{ij}, κ_{ij} — components of deformation, ε_{ijk} — alternating symbol, F_i — components of body force, M_i — components of body couple; the comma denotes partial derivation with respect to the variable x_i , (1) and (2) denote the derivation with respect to time once, respectively twice; $\rho, I_{ij}, A_{ijkl}, B_{ijkl}, C_{ijkl}$ — are functions characterizing the respective material, and $a_i, b_i, c_i, d_i, \tilde{u}_i, \tilde{\varphi}_i, \tilde{t}_i, \tilde{m}_i$ — are prescribed functions. Consider that in the initial state the following conditions are fulfilled

$$(1.7) \quad u_i(x, t) = \varphi_i(x, t) \quad \text{on } \bar{\Omega} \times (-\infty, 0)$$

from where we have

$$(1.8) \quad \gamma_{ij}(x, t) = \kappa_{ij}(x, t) = t_{ij}(x, t) = m_{ij}(x, t) = 0 \quad \text{on } \bar{\Omega} \times (-\infty, 0).$$

Constitutive equations can also be expressed as

$$(1.9) \quad t_{ij} = \bar{t}_{ij} + [A_{ijkl} * \gamma_{kl} + B_{ijkl} * \kappa_{kl}]^{(1)}, \quad m_{ij} = \bar{m}_{ij} + [B_{klij} * \gamma_{kl} + C_{ijkl} * \kappa_{kl}]^{(1)},$$

where

$$(1.10) \quad \begin{aligned} \bar{t}_{ij} &= \int_0^\infty [A_{ijkl}^{(1)}(x, t+\tau) \gamma_{kl}^0(x, -\tau) + B_{ijkl}^{(1)}(x, t+\tau) \kappa_{kl}^0(x, -\tau)] d\tau, \\ \bar{m}_{ij} &= \int_0^\infty [B_{klij}(x, t+\tau) \gamma_{kl}^0(x, -\tau) + C_{ijkl}^{(1)}(x, t+\tau) \kappa_{kl}^0(x, -\tau)] d\tau. \end{aligned}$$

or, taking into account (1.8)

$$(1.11) \quad \begin{aligned} t_{ij} &= \overset{\circ}{A}_{ij k_l}(x) \gamma_{k_l}(x, t) + \overset{\circ}{B}_{ij k_l}(x) \kappa_{k_l}(x, t) + A_{ij k_l}^{(1)} * \gamma_{k_l} + B_{ij k_l}^{(1)} * \kappa_{k_l}, \\ m_{ij} &= \overset{\circ}{B}_{k_l ij}(x) \gamma_{k_l}(x, t) + \overset{\circ}{C}_{ij k_l}(x) \kappa_{k_l}(x, t) + B_{k_l ij}^{(1)} * \gamma_{k_l} + C_{ij k_l}^{(1)} * \kappa_{k_l}, \end{aligned}$$

where

$$\overset{\circ}{A}_{ij k_l}(x) = A_{ij k_l}(x, 0), \quad \overset{\circ}{B}_{ij k_l}(x) = B_{ij k_l}(x, 0), \quad \overset{\circ}{C}_{ij k_l}(x) = C_{ij k_l}(x, 0).$$

2. Variational Theorems. We define as an admissible state $S = \{u_i, \varphi_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}\}$ the ordered assembly of functions $u_i, \varphi_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}$ defined on $\bar{\Omega} \times (0, \infty)$ with the properties

$$u_i \in C^{1,2}, \quad \varphi_i \in C^{1,2}, \quad \gamma_{ij} \in C^{0,0}, \quad \kappa_{ij} \in C^{0,0}, \quad t_{ij} \in C^{1,0}, \quad m_{ij} \in C^{1,0}.$$

If we define the addition of the admissible states and the multiplication of an admissible state by a scalar as

$$S + S' = \{u_i + u'_i, \varphi_i + \varphi'_i, \gamma_{ij} + \gamma'_{ij}, \kappa_{ij} + \kappa'_{ij}, t_{ij} + t'_{ij}, m_{ij} + m'_{ij}\},$$

$$\alpha S = \{\alpha u_i, \alpha \varphi_i, \alpha \gamma_{ij}, \alpha \kappa_{ij}, \alpha t_{ij}, \alpha m_{ij}\}, \quad \forall \alpha \in \mathbb{R},$$

we find out that the set of all admissible states is a linear space. We shall define a kinematic admissible state as an admissible state satisfying equations (1.2), (1.9) as well as the conditions imposed on the boundary part Σ_1 and Σ_2 . We call solution of the problem an admissible state satisfying equations (1.1)–(1.3), initial conditions (1.5) and boundary conditions (1.6).

For simplicity sake, we shall neglect the regularity hypotheses which will be necessary later on.

Theorem 2.1 [4]. *The functions $u_i, \varphi_i, t_{ij}, m_{ij}$ – satisfy the equations (1.1) and the initial conditions (1.5) if and only if*

$$(2.1) \quad \begin{aligned} g * t_{ij,j} + f_i &= \rho u_i, \\ g * (m_{j,i,j} + \varepsilon_{ijk} t_{jk}) + g_i &= I_{ij} \varphi_i, \quad \text{on } \Omega \times (0, \infty), \end{aligned}$$

where

$$(2.2) \quad \begin{aligned} g(t) &= t, \\ f_i &= g * F_i + \rho(t b_i + a_i), \\ g_i &= g * M_i + I_{ij}(t d_j + c_j), \quad t > 0. \end{aligned}$$

Thus we have an alternative characterization of the solution to the mixed problem in which the initial conditions are incorporated into one of the field equations.

Theorem 2.2 [4]. *An admissible state S can be considered solution of the mixed problem if and only if satisfies equations (2.1), (1.2), (1.3), (1.6). Replacing (1.2) in (1.9) and the result in (2.1) we obtain*

$$(2.3) \quad Au = f + \bar{f},$$

where

$$(2.4) \quad A_i u = f_i + \bar{f}_i, \quad A_{i+3} u = g_i + \bar{g}_i,$$

$$(2.5) \quad \begin{aligned} A_i u &= \rho u_i - l * [A_{jikl} * (u_{l,k} + \varepsilon_{lkm} \varphi_m) + B_{jlik} * \varphi_{l,k}]_{,j}, \\ A_{i+j} u &= I_{ij} \varphi_j - l * [A_{jkmn} * (u_{n,m} + \varepsilon_{nmp} \varphi_p) + B_{jkmn} * \varphi_{n,m}] \varepsilon_{ijk} - \\ &\quad - l * [B_{klij} * (u_{l,k} + \varepsilon_{lkm} \varphi_m) + C_{jlik} * \varphi_{l,k}]_{,j}, \end{aligned}$$

$$(2.6) \quad l(t) = 1, \quad t > 0,$$

$$(2.7) \quad f = (f_1, f_2, f_3, g_1, g_2, g_3), \quad \bar{f} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{g}_1, \bar{g}_2, \bar{g}_3),$$

$$(2.8) \quad \bar{f}_i = g * \bar{t}_{ji,j}, \quad \bar{g}_i = g * (\bar{m}_{ji,j} + \varepsilon_{ijk} \bar{t}_{jk}).$$

We have also used the fact that

$$(2.9) \quad g * h^{(1)} = l * h,$$

if $h(0) = 0$, g and h are defined by (2.2) and (2.6).

If in the reciprocity relation from [2] we introduce the notations

$$(2.10) \quad \begin{aligned} u &= (u_i^1, \varphi_i^1), \quad v = (u_i^2, \varphi_i^2), \quad t(u) = (t_i^1, m_i^1), \quad t(v) = (t_i^2, m_i^2), \\ \bar{t}(u) &= (\bar{t}_i^1, \bar{m}_i^1), \quad \bar{t}(v) = (\bar{t}_i^2, \bar{m}_i^2), \end{aligned}$$

then we write

$$(2.11) \quad \int_{\Omega} (v * A u - u * A v) d\Omega = \int_{\Sigma} g * [u * (t(v) + \bar{t}(v)) - v * (t(u) + \bar{t}(u))] d\sigma,$$

where

$$(2.12) \quad \bar{t}_i^\alpha = -\bar{t}_{ji}^\alpha n_j, \quad \bar{m}_i^\alpha = -\bar{m}_{ji}^\alpha n_j.$$

Introducing the convolution scalar product of f and g functions, defined by [4]

$$(2.13) \quad (f \otimes g) = \int_{\Omega} f * g d\Omega,$$

relation (2.11) becomes

$$(2.14) \quad (A u \otimes v) - (u \otimes A v) = \int_{\Sigma} g * [u * (t(v) + \bar{t}(v)) - v * (t(u) + \bar{t}(u))] d\sigma.$$

We consider the case when boundary conditions are homogeneous. From (2.14) it results that operator A is symmetric in convolution. Consequently, for the functional

$$(2.15) \quad F(u) = (A u \otimes u) - 2(u \otimes f)$$

we have

$$(2.16) \quad \delta F(u) = 0, \quad \text{on } D_A$$

if and only if $u \in D_A$ satisfies equation (2.3), D_A being the definition domain of operator A .

it is observed that

$$(2.17) \quad (Au \otimes u) = \int_{\Omega} l^*(A_{ij k_l} \gamma_{ij} * \gamma_{k_l} + 2B_{ij k_l} \gamma_{ij} * \kappa_{k_l} + C_{ij k_l} \kappa_{ij} * \kappa_{k_l}) d\Omega + \\ + \int_{\Omega} (\rho u_i * u_i + I_{ij} \varphi_i * \varphi_j) d\Omega.$$

Taking into account (2.15) and (2.17) we get to

$$(2.18) \quad \Gamma(u) = \int_{\Omega} l^*(A_{ij k_l} \gamma_{ij} * \gamma_{k_l} + 2B_{ij k_l} \gamma_{ij} * \kappa_{k_l} + C_{ij k_l} \kappa_{ij} * \kappa_{k_l}) d\Omega + \\ + \int_{\Omega} (\rho u_i * u_i + I_{ij} \varphi_i * \varphi_j) d\Omega - 2 \int_{\Omega} (f_i * u_i + g_i * \varphi_i) d\Omega,$$

where from it results

Theorem 2.3 Let $\mathcal{M} \subset D_A$ be the vector set $u = (u_i, \varphi_i)$ satisfying conditions (1.6) in a homogeneous form. Then

$$(2.19) \quad \delta \Gamma(u) = 0, \quad \text{on } \mathcal{M},$$

if and only if $u \in \mathcal{M}$ is a solution of a mixed problem.

We consider the case of inhomogeneous boundary conditions. Let $K \subset D_A$ be the vector set $u = (u_i, \varphi_i)$ satisfying conditions

$$(2.20) \quad u_i = \tilde{u}_i \text{ on } \Sigma_1 \times (0, \infty), \quad \varphi_i = \tilde{\varphi}_i \text{ on } \Sigma_3 \times (0, \infty).$$

In this case we get the following functional

$$(2.21) \quad \Lambda(u) = \int_{\Omega} l^*(A_{ij k_l} \gamma_{ij} * \gamma_{k_l} + 2B_{ij k_l} \gamma_{ij} * \kappa_{k_l} + C_{ij k_l} \kappa_{ij} * \kappa_{k_l}) d\Omega + \\ + \int_{\Omega} (\rho u_i * u_i + I_{ij} \varphi_i * \varphi_j) d\Omega - 2 \int_{\Omega} (f_i * u_i + g_i * \varphi_i) d\Omega - \\ - 2 \int_{\Sigma_2} g * \tilde{l}_i * u_i d\sigma - 2 \int_{\Sigma_4} g * \tilde{m}_i * \varphi_i d\sigma$$

and have

Theorem 2.4. The vector $u \in K$ is a solution of a mixed problem if and only if

$$(2.22) \quad \delta \Lambda(u) = 0, \quad \text{on } K.$$

Consider $S = \{u_i, \varphi_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}\}$ a kinematic admissible state and the functional

$$\begin{aligned}
 \psi_t(S) = & \frac{1}{2} \int_{\Omega} l * A_{ijk_l} * \gamma_{ij} * \gamma_{kl} d\Omega + \int_{\Omega} l * B_{ijk_l} * \gamma_{ij} * \kappa_{kl} d\Omega + \\
 (2.23) \quad & + \frac{1}{2} \int_{\Omega} l * C_{ijk_l} * \kappa_{ij} * \kappa_{kl} d\Omega - \int_{\Omega} g * \bar{t}_{ij} * \gamma_{ij} d\Omega - \int_{\Omega} g * \bar{m}_{ij} * \kappa_{ij} d\Omega - \\
 & - \int_{\Omega} g * F_i * u_i d\Omega - \int_{\Omega} g * M_i * \varphi_i d\Omega - \int_{\Sigma_2} g * \tilde{t}_i * u_i d\sigma - \int_{\Sigma_4} g * \tilde{m}_i * \varphi_i d\sigma,
 \end{aligned}$$

defined on set K of the kinematic admissible state.

Theorem 2.5. *If S is a solution to the mixed problem and*

$$\begin{aligned}
 (2.24) \quad & A_{ijk_l} * \eta_{ij} * \eta_{kl} + 2B_{ijk_l} * \eta_{ij} * \eta_{kl} + C_{ijk_l} * \eta_{ij} * \eta_{kl} \geq 0, \quad \forall \eta_{ij} \neq 0, \\
 & \text{then}
 \end{aligned}$$

$$(2.25) \quad \psi_t(S) \leq \psi_t(S^*), \quad \forall S^* \in K.$$

Proof. Consider $S, S^* \in K$ and $S' = S^* - S$. We have

$$(2.26) \quad \gamma'_{ij} = u'_{j,i} + \varepsilon_{jim} \varphi'_m, \quad \kappa'_{ij} = \varphi'_{j,i},$$

$$\begin{aligned}
 (2.27) \quad & t'_{ij} = \bar{t}_{ij} + [A_{ijk_l} * \gamma'_{kl} + B_{ijk_l} * \kappa'_{kl}]^{(1)}, \quad m'_{ij} = \bar{m}_{ij} + [B_{klij} * \gamma'_{kl} + C_{ijk_l} * \kappa'_{kl}]^{(1)}, \\
 & \text{on } \Omega
 \end{aligned}$$

and

$$(2.28) \quad u'_i = 0 \text{ on } \Sigma_1, \quad \varphi'_i = 0 \text{ on } \Sigma_3.$$

Taking into account (2.8) and (1.6) the functional $\psi_t(S^*)$ can be put in the form

$$\begin{aligned}
 \psi_t(S^*) = & \psi_t(S) + \frac{1}{2} \int_{\Omega} l * A_{ijk_l} * \gamma'_{ij} * \gamma'_{kl} d\Omega + \int_{\Omega} l * B_{ijk_l} * \gamma'_{ij} * \kappa'_{kl} d\Omega + \\
 (2.29) \quad & + \frac{1}{2} \int_{\Omega} l * C_{ijk_l} * \kappa'_{ij} * \kappa'_{kl} d\Omega + \int_{\Omega} g * t'_{ij} * \gamma'_{ij} d\Omega + \int_{\Omega} g * m'_{ij} * \kappa'_{ij} d\Omega - \\
 & - \int_{\Omega} g * F_i * u'_i d\Omega - \int_{\Omega} g * M_i * \varphi'_i d\Omega - \int_{\Sigma_2} g * \tilde{t}_i * u'_i d\sigma - \int_{\Sigma_4} g * \tilde{m}_i * \varphi'_i d\sigma.
 \end{aligned}$$

Replacing (2.26) in (2.29) and applying the divergence theorem, taking into account (2.28) and grouping correspondingly, we get

$$\begin{aligned}
 \psi_t(S^*) = & \psi_t(S) + \frac{1}{2} \int_{\Omega} l * A_{ijk_l} * \gamma'_{ij} * \gamma'_{kl} d\Omega + \int_{\Omega} l * B_{ijk_l} * \gamma'_{ij} * \kappa'_{kl} d\Omega + \\
 (2.30) \quad & + \frac{1}{2} \int_{\Omega} l * C_{ijk_l} * \kappa'_{ij} * \kappa'_{kl} d\Omega - \int_{\Omega} g * (t_{ji,j} + F_i) * u'_i d\Omega - \int_{\Omega} g * (m_{ji,j} + \varepsilon_{ijk} t_{jk} + \\
 & + M_i) * \varphi'_i d\Omega + \int_{\Sigma_1} g * (t_{ji} \eta_j - \tilde{t}_i) * u'_i d\sigma + \int_{\Sigma_4} g * (m_{ji} n_j - \tilde{m}_i) * \varphi'_i d\sigma,
 \end{aligned}$$

S being a solution of the mixed problem in the case of viscoelastic equilibrium, from (2.20) it results

$$(2.31) \quad \psi_t(S^*) - \psi_t(S) = \frac{1}{2} \int_{\Omega} l * (A_{ij k l} * \gamma'_{ij} * \gamma'_{kl} + 2B_{ij k l} * \gamma'_{ij} * \kappa'_{kl} + C_{ij k l} * \kappa'_{ij} * \kappa'_{kl}) d\Omega,$$

thus, respecting the condition (2.24) it can be written

$$(2.32) \quad \psi_t(S) \leq \psi_t(S^*),$$

$$(2.33) \quad \psi_t(S) = \psi_t(S^*) \Leftrightarrow \gamma_{ij}^* = \gamma_{ij}^* - \gamma_{ij} = 0, \quad \kappa_{ij}^* = \kappa_{ij}^* - \kappa_{ij} = 0.$$

Consider $S = \{u_i, \varphi_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}\}$ an admissible state and the functional

$$(2.34) \quad \begin{aligned} J_t(S) = & \frac{1}{2} [\rho u_i^{(1)}, u_i^{(1)}] + \frac{1}{2} [I_{ij} \varphi_i^{(1)}, \varphi_j^{(1)}] + [t_{ji}, u_{i,j} + \varepsilon_{mji} \varphi_m - \gamma_{ji}] - \\ & - [F_i, u_i] + [m_{ij}, \varphi_{i,j} - \kappa_{ji}] - [M_i, \varphi_i] + \frac{1}{2} [A_{jikl}^{(1)} * \gamma_{kl} + \hat{A}_{jikl} \gamma_{kl}, \kappa_{ji}] + \\ & + \frac{1}{2} [B_{jikl}^{(1)} * \kappa_{kl} + \hat{B}_{jikl} \kappa_{kl}, \gamma_{ji}] + \frac{1}{2} [B_{klji}^{(1)} * \gamma_{kl} + \hat{B}_{klji} \gamma_{kl}, \kappa_{ji}] + \\ & + \frac{1}{2} [C_{jikl}^{(1)} * \kappa_{kl} + \hat{C}_{jikl} \kappa_{kl}, \kappa_{ji}] - [t_i, u_i - \tilde{u}_i]_{\Sigma_1} - [\tilde{t}_i, u_i]_{\Sigma_2} - \\ & - [m_i, \tilde{\varphi}_i - \varphi_i]_{\Sigma_3} - [\tilde{m}_i, \varphi_i]_{\Sigma_4} + [\rho(u_i^{(1)} - b_i), u_i]_0 - [\rho(u_i - 2a_i), u_i^{(1)}]_0 + \\ & + [I_{ij}(\varphi_i^{(1)} - d_i), \varphi_j]_0 - [I_{ij}(\varphi_i - 2c_i), \varphi_i^{(1)}]_0, \end{aligned}$$

defined on the K set of the admissible states, where

$$(2.35) \quad \begin{aligned} [f, h] &= \int_{\Omega} \int_0^t f(x, \tau) h(x, t - \tau) d\tau d\Omega, \\ [f, h]_0 &= \int_{\Omega} \{f(x, \tau) h(x, t - \tau)\}_{\tau=0}^t d\Omega, \\ [f, h]_{\Sigma} &= \int_{\Sigma} \int_0^t f(x, \tau) h(x, t - \tau) d\tau d\sigma. \end{aligned}$$

Theorem 2.6. Let $S = \{u_i, \varphi_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}\} \in K$ and for $t \in [0, \infty)$ define the functional $J_t(S)$ on K through (2.34). Then

$$(2.36) \quad \delta J_t(S) = 0$$

if and only if S is a solution of the mixed problem.

Proof. Consider $S = \{u_i, \varphi_i, \gamma_{ij}, \kappa_{ij}, t_{ij}, m_{ij}\}$ and $S' = \{u'_i, \varphi'_i, \gamma'_{ij}, \kappa'_{ij}, t'_{ij}, m'_{ij}\}$ two admissible states. It is obvious $S + \alpha S' \in K$ for every real α , ($\alpha \in \mathbb{R}$). If we can form the expression

$$\delta_s' J_t(S) = \frac{d}{d\alpha} J_t(S + \alpha S') \Big|_{\alpha=0},$$

we can deduce that

$$\begin{aligned}
\delta_S J_t(S) = & [\rho u_i^{(1)}, u_i^{(1)}] + [I_{ij} \varphi_i^{(1)}, \varphi_j^{(1)}] + [t_{ji}, u'_{i,j} + \varepsilon_{mji} \varphi'_m - \gamma'_{ji}] + \\
& + [u_{i,j} + \varepsilon_{mji} \varphi_m - \gamma_{ji}, t'_{ji}] + [m_{ji}, \varphi'_{i,j} - x'_{ji}] + [\varphi_{i,j} - x_{ji}, m'_{ji}] - \\
& - [F_i, u'_i] - [M_i, \varphi'_i] + [A_{ji}^{(1)} k_l * \gamma_{kl} + B_{ji}^{(1)} k_l * x_{kl} + \dot{A}_{ji} k_l \gamma_{kl} + \\
(2.37) \quad & + \dot{B}_{ji} k_l x_{kl}, \gamma'_{ji}] + [B_{kl}^{(1)} j_i * \gamma_{kl} + C_{ji}^{(1)} k_l * x_{kl} + \dot{B}_{kl} j_i \gamma_{kl} + \dot{C}_{ji} k_l x_{kl}, x'_{ji}] - \\
& - [t_i, u'_i]_{\Sigma_1} - [u_i - \tilde{u}_i, t'_i]_{\Sigma_1} - [\tilde{t}_i, u'_i]_{\Sigma_2} - [\varphi_i - \varphi_i, m'_i]_{\Sigma_3} - [\tilde{m}_i, \varphi'_i]_{\Sigma_4} + \\
& + [m_i, \varphi'_i]_{\Sigma_5} + [\rho u_i^{(1)} - h_i, u'_i]_0 + [\rho u_i^{(1)}, u_i]_0 - [\rho(u_i - 2a_i), u_i^{(1)}]_0 - \\
& - [\rho u'_i, u_i^{(1)}]_0 + [I_{ij}(\varphi_i^{(1)} - d_i), \varphi'_j]_0 + [I_{ij} \varphi_i^{(1)}, \varphi_j]_0 - \\
& - [I_{ij}(\varphi_i - 2c_i), \varphi_j^{(1)}]_0 - [I_{ij} \varphi'_i, \varphi_j^{(1)}]_0.
\end{aligned}$$

Integrating by parts and taking into account the divergence theorem we get

$$\begin{aligned}
\delta_S J_t(S) = & [u_{i,j} + \varepsilon_{mji} \varphi_m - \gamma_{ji}, t'_{ji}] + [\varphi_{i,j} - x_{ji}, m'_{ji}] - \\
& - [t_{ji,j} + F_i - \rho u_i^{(2)}, u'_i] - [m_{ji,j} + \varepsilon_{ijm} t_{jm} + M_i - I_{ij} \varphi_i^{(2)}, \varphi'_i] + \\
(2.38) \quad & + [A_{ji}^{(1)} k_l * \gamma_{kl} + B_{ji}^{(1)} k_l * x_{kl} + \dot{A}_{ji} k_l \gamma_{kl} + \dot{B}_{ji} k_l x_{kl} - t_{ji}, \gamma'_{ji}] + \\
& + [B_{kl}^{(1)} j_i * \gamma_{kl} + C_{ji}^{(1)} k_l * x_{kl} + \dot{B}_{kl} j_i \gamma_{kl} + \dot{C}_{ji} k_l x_{kl} - m_{ji}, x'_{ji}] - \\
& - [u_i - \tilde{u}_i, t'_i]_{\Sigma_1} - [\tilde{t}_i - t_i, u'_i]_{\Sigma_2} - [\varphi_i - \tilde{\varphi}_i, m'_i]_{\Sigma_3} - [\tilde{m}_i - m_i, \varphi'_i]_{\Sigma_4} + \\
& + [\rho(u_i^{(1)} - h_i), u'_i]_0 - 2[\rho(u_i - j_i), u_i^{(1)}]_0 + [I_{ij}(\varphi_i^{(1)} - d_i), \varphi_j^{(1)}]_0 - \\
& - 2[I_{ij}(\varphi_i - c_i), \varphi_j^{(1)}]_0.
\end{aligned}$$

If S is a solution of the mixed problem, then from (2.38) it results

$$(2.39) \quad \delta_{S'} J_t(S) = 0, \quad \forall S' \in K$$

and thus (2.36) takes place ($(\delta_{S'} J_t(S))$ being linear in S').

Inversely, we suppose that relation (2.36) takes place, that implying (2.39). It is obvious that $\dot{S} = \{0, 0, 0, 0, 0, 0\}$ and thus we can select $S' = \{u'_i, 0, 0, 0, 0, 0\}$ so that u'_i , together with all of its space derivatives, vanish on $\Sigma \times [0, \infty)$. Thus, (2.38) and (2.39) with the selection made implies

$$(2.40) \quad \int_{\Omega} \{t_{ji,j} + F_i - \rho u_i^{(2)}\} * u'_i d\Omega = 0, \quad t \geq 0$$

and on the basis of the fundamental lemma of the calculus of variations [10] it results

$$(2.41) \quad t_{ji,j} + F_i - \rho u_i^{(2)} = 0.$$

Consider the same choice of S' but require now that u'_i vanish on $\Sigma_1 \times [0, \infty)$. Then (2.38), (2.39), (2.41) we have

$$(2.42) \quad \int_{\Sigma_1} \{\tilde{t}_i - t_i\} * u'_i d\sigma = 0,$$

where from, generalizing the fundamental lemma, it results

$$(2.43) \quad t_i = \tilde{t}_i \quad \text{on } \Sigma_2 \times [0, \infty).$$

Now let $S' = \{0, \varphi'_i, 0, 0, 0, 0\}$ and suppose φ'_i and all of its space derivatives vanish on $\Sigma \times [0, \infty)$. Then by (2.38), (2.39) and the fundamental lemma, we get

$$(2.44) \quad m_{ij,j} + \varepsilon_{ijm} t_{jm} + M_i - I_{ij} \varphi_j^{(2)} = 0.$$

Next let $S' = \{0, \varphi'_i, 0, 0, 0, 0\}$ but this time require merely that vanish on $\Sigma \times [0, \infty)$. Then (2.38), (2.39), (2.43), (2.44) and the lemma of the variational calculus imply

$$(2.45) \quad m_i = \tilde{m}_i \quad \text{on } \Sigma_4 \times [0, \infty).$$

Continuing in this way there results the proof of the theorem.

Received September 25, 1989

Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Ieșan C. M., *Sur quelques théorèmes généraux dans la théorie linéaire des milieux élastiques avec microstructure*. C. Rend. Acad. Sc. Paris, **270** (1970).
2. Ieșan D., *Sur la théorie de la viscoélasticité micropolaire*. C. Rend. Acad. Sc. Paris, **270** (1970).
3. Ieșan D., *On Some Reciprocity Theorems and Variational Theorems in Linear Dynamic Theories of Continuum Mechanics*. Memorie dell'Acad. Sci. Torino, cl. Sci. Fis. Mat. Nat., Serie 4, 17 (1974).
4. Ieșan D., *Unele probleme fundamentale ale teoriei liniare a mediilor elastice de tip Cosserat*. „Probleme actuale în Mecanica Solidelor”, vol. II, Ed. Acad. R.S.R., Buc., 1975.
5. Ieșan D., *Teoria termoelasticității*. Ed. Acad. R.S.R., Buc., 1975.
6. Gurtin M. E., *Variational Principles in the Linear Theory of Viscoelasticity*. Arch. Rational Mech. Anal., **3** (1963).
7. Reddy N. J., *Modified Gurtin's Variational Principles in the Linear Dynamic Theory of Viscoelasticity*. Int. J. Solids Structures, **12** (1976).
8. Kline K. A., *Theorems of Ignaczak Type for Elastic Materials with Microstructure*. Bull. Acad. Polon. Sci., Ser. Sci. Techn., **18** (1970).
9. Kline K. A., *Variational Principles in the Linear Theory of Viscoelastic Media with Microstructure*. Quart. Appl. Math., **28**, 1 (1970).
10. Leitman L. M., *Variational Principles in the Linear Dynamic Theory of Viscoelasticity*. Quart. Appl. Math., **24**, 1 (1966).
11. Manole D., *A Reciprocity Theorem and a Variational Theorem for Elastic Media with Microstructure*. Bul. Inst. Polit. Iași, **XXXII(XXXVI)**, 1–4, s. I (1986).
12. Manole D., *Reciprocity and Variational Theorems in the Linear Theory of Viscoelastic Media with Microstructure*. Bul. Inst. Polit. Iași, Cercetări matematice, 1985.

TEOREME VARIAȚIONALE ÎN TEORIA VISCOELASTICITĂȚII LINIARE MICROPOLARE

(Rezumat)

Se prezintă câteva teoreme variaționale în teoria viscoelasticității liniare micropolare.

THE UNIVERSITY OF CHICAGO

PHYSICS DEPARTMENT

CHICAGO, ILLINOIS

OFFICE OF THE DEAN

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

CHICAGO, ILLINOIS

MULTIPLE TIME SCALE METHOD AND THE CONSTRUCTION OF A „SUPERCLOCK“

BY

CORNELIU D. CIUBOTARIU and ILIE BURDUJAN

1. Introduction. To incorporate the disparity between various time scale of physical phenomena (e. g. rapid oscillation and growth rate [1], oscillation and rate at which energy is fed into the oscillations [2] etc.) in an expansion procedure, we arbitrarily extend the number of time variable t to many successive time variables (scales) $\tau_0 \equiv t$, $\tau_1 = \varepsilon \tau_0$, $\tau_2 = \varepsilon^2 \tau_0$, The unknown dependent variables f_α (e. g. physical quantities) become functions of these new multiple variables, and can be expanded in powers of the small parameter ε ($0 < \varepsilon < 1$) according to

$$(1.1) \quad f_\alpha(\tau_0, \tau_1, \dots) = \sum_{n=0}^{\infty} \varepsilon^n f_{\alpha(n)}(\tau_0, \tau_1, \dots).$$

Operationally we treat $\tau_0, \tau_1, \tau_2, \dots$ as independent variables, and expand the time derivatives in the equation for f_α by putting

$$(1.2) \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \dots$$

The small parameter ε then measures the ratio of the fast time scale to the slow one.

The usual procedure in the multiple-timing method is to assume that the solution has the form (1.1), substitute this in the governing system of differential equations, and equate to zero the coefficients of the successive powers of ε . In other words, our system splits into a sequence of equations in which may occur secular resonant terms. One normally then has to impose further restrictions on the $f_{\alpha(n)}$ in order that the $f_{\alpha(n+1)}$ remain bounded, i. e. so that each term has its assumed magnitude [3]. Since the number of time variables has been increased, $f_{\alpha(n)}$ ($n=0, 1, 2, \dots$) can be considered as „generalized“ or „extended“ functions which include the solutions of physical interest, and provide us with additional freedom in the perturbation analysis. Then, the equations for $f_{\alpha(n)}$ can be solved step by step (order by order) by cancelling the time secularities (proportional to t , εt , $\varepsilon^2 t^2$, etc.) which may occur in the solution for $f_{\alpha(n)}$ at each stage of the calculation. We can thus calculate the successive approximations which are unfortunately valid and remain bounded. In each application the removal of time secularities on a

fast time scale leads to a nonlinear differential equation which determines the time development of the system on a slower time scale. In order to apply this method, for each particular physical situation, it is necessary to define an ordering scheme which allows to determine the relevant small parameter ε well adapted for the considered problem; in many cases, this parameter characterizes both the nonlinearity and the dissipativity of the system.

This type of perturbation theory which is free of secular terms is known as multiple time scale method of Krylov-Bogoliubov-Frieman and Sándri [4], [7]. From a mathematical point of view the method consists in the construction of an extension $\tilde{f}$ of the unknown function f to an extended domain which must contain a curve Γ such that the restriction of $\tilde{f}$ to Γ will be f . The construction of the extension $\tilde{f}$ is not unique ("effective") because it depends on three elements: dimension of the domain extension, choosing of Γ , and choosing of the extension itself.

In this paper we propose to demonstrate an induction theorem which expresses the fact that all extensions of f are "induced" by extension of a simple mapping of the original domain. Furthermore, we present a natural way of introduction of a superclock as a pure mathematical necessity.

2. The Extension of Solutions of the Autonomous Differential Equations. Consider the solution $f(t)$ of the autonomous differential equation

$$(2.1) \quad \frac{df}{dt} = G[f],$$

with the initial condition

$$(2.2) \quad f(0) = a$$

and the "trajectory" $\Gamma \subset \mathbb{R}^{N+1}$ defined by the autonomous system

$$(2.3) \quad \frac{d\tau_0}{dt} = g_0(\tau_0, \tau_1, \dots, \tau_N), \dots, \frac{d\tau_N}{dt} = g_N(\tau_0, \tau_1, \dots, \tau_N)$$

with the initial conditions

$$(2.4) \quad \tau_i(0) = a_i, \quad i = 0, 1, 2, \dots, N.$$

The solution

$$(2.5) \quad \tau_i = \tau_i(t), \quad i = 0, 1, 2, \dots, N$$

represents the finite equations for curve Γ .

We say that $\tilde{f}: \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ is an extension of f on Γ if and only if

$$(2.6) \quad \tilde{f}(\tau_0(t), \tau_1(t), \dots, \tau_N(t)) = f(t)$$

and thus

$$(2.7) \quad \frac{df}{dt} = \sum_{i=0}^N \left[\frac{\partial \tilde{f}}{\partial \tau_i} g_i(\tau_0, \tau_1, \dots, \tau_N) \right]_{\Gamma}.$$

The Eq. (2.7) suggests an approach for the construction of an extension. We consider the equation

$$(2.8) \quad \sum_{i=0}^N \frac{\partial \tilde{f}}{\partial \tau_i} g_i(\tau_0, \tau_1, \dots, \tau_N) = G[\tilde{f}],$$

and we choose a solution which satisfies the condition

$$(2.9) \quad \tilde{f}(a_0, a_1, \dots, a_N) = f(0).$$

This solution is an extension of f on Γ . Indeed, if we consider the function $g(t) = \tilde{f}(\tau_0(t), \tau_1(t), \dots, \tau_N(t))$, we have

$$(2.10) \quad \frac{dg}{dt} = \sum_{n=0}^N \left[\frac{\partial \tilde{f}}{\partial \tau_n} g_n \right]_{\Gamma} = G(\tilde{f}(\tau_0(t), \dots, \tau_N(t))) = G(g(t)),$$

and

$$(2.11) \quad g(0) = \tilde{f}(a_0, a_1, \dots, a_N) = f(0).$$

Hence we must have

$$(2.12) \quad g(t) \equiv f(t)$$

and thus $\tilde{f}$ is an extension of f .

3. The Case of Functions Depending on a Small Parameter. We apply the method of Sec. 2 to functions $f(\varepsilon, t)$, which are solutions of the autonomous differential equation

$$(3.1) \quad \frac{df(\varepsilon, t)}{dt} = G[\varepsilon, f(\varepsilon, t)],$$

with the condition

$$(3.2) \quad f(\varepsilon, 0) = a(\varepsilon).$$

Consider the curves $\Gamma(\varepsilon)$, $\tau_i = \tau_i(\varepsilon, t)$, ($i=0, 1, 2, \dots, N$) that satisfy the system

$$(3.3) \quad \frac{d\tau_0}{dt} = g_0(\varepsilon, \tau), \dots, \frac{d\tau_N}{dt} = g_N(\varepsilon, \tau),$$

with the initial conditions

$$(3.4) \quad \tau_i(\varepsilon, 0) = a_i(\varepsilon), \quad i=0, 1, 2, \dots, N,$$

where

$$(3.5) \quad \tau = (\tau_0, \tau_1, \dots, \tau_N).$$

With the approach of Sec. 2 we choose the extension $\tilde{f}(\varepsilon, \tau)$ as a solution of the equation

$$(3.6) \quad \sum_{n=0}^N \frac{\partial \tilde{f}(\varepsilon, \tau)}{\partial \tau_n} g_n(\varepsilon, \tau) = G(\varepsilon, \tilde{f}(\varepsilon, \tau))$$

with the conditions

$$(3.7) \quad \tilde{f}(\varepsilon, a_0(\varepsilon), a_1(\varepsilon), \dots, a_N(\varepsilon)) = f(\varepsilon, 0).$$

We see that the extension $\tilde{f}$ which correspond to a given function f is not unique, even when a definite curve Γ has been choosen. Indeed, the Eq. (3.6) is a partial differential equation which have infinitely many solutions corresponding to hypersurfaces in $\mathbb{R}^{N+1}$ with the same common point $(a_0(\varepsilon), a_1(\varepsilon), \dots, a_N(\varepsilon), f(0))$. We note that the curves Γ , given by the Eqs. (3.3) and (3.4), are the characteristic lines of the partial differential equation (3.6).

The advantage of dealing with the extended functions $\tilde{f}$ is the possibility of choosing $\tilde{f}$ with simpler and smoother dependence on the parameter ε than that offered by f itself [7]. Thus, corresponding to an f which is not even an analytic (entire) function of the variable εt one can always choose an $\tilde{f}$ that is an analytic function of the variable ε . For example, $f = \exp(-t/\varepsilon)$ and $\tilde{f} = \exp(-\tau_{-1})$ with $\tau_{-1} = t/\varepsilon$ [7]. In other words, the extension $\tilde{f}$ is designed to exploit to the fullest the presence of a small parameter whenever one its appears in a problem. To see this, we expand the functions $\tilde{f}$, g_n and G in powers of ε and demand that the expansion be asymptotic in ε , that is

$$(3.8) \quad \tilde{f}(\varepsilon, \tau) = \tilde{f}_0(\tau) + O(\varepsilon) = \tilde{f}_0(\tau) + \varepsilon \tilde{f}_1(\tau) + O(\varepsilon^2) + \dots,$$

$$(3.9) \quad g_n(\varepsilon, \tau) = g_{n0}(\tau) + \varepsilon g_{n1}(\tau) + \dots,$$

$$(3.10) \quad G(\varepsilon, \tilde{f}(\varepsilon, \tau)) = G(0, \tilde{f}_0(\tau)) + \varepsilon \left[\frac{\partial G}{\partial \varepsilon}(0, \tilde{f}_0(\tau)) + \frac{\partial G}{\partial \tilde{f}}(0, \tilde{f}_0(\tau)) \tilde{f}_1(\tau) \right] + \dots$$

(here $\tilde{f}_i(\tau) = \tilde{f}_i(0, \tau)$ and $g_{ni}(\tau) = g_{ni}(0, \tau)$, $i=0, 1, \dots$).

Substituting the Eqs. (3.8)–(3.10) into (3.6) we find, equating powers of ε , that

$$(3.11) \quad \sum_{n=0}^N \frac{\partial \tilde{f}_0(\tau)}{\partial \tau_n} g_{n0}(\tau) = G(0, \tilde{f}_0(\tau)),$$

$$(3.12) \quad \sum_{n=0}^N \frac{\partial \tilde{f}_1(\tau)}{\partial \tau_n} g_{n0}(\tau) + \sum_{n=0}^N \frac{\partial \tilde{f}_0(\tau)}{\partial \tau_n} g_{n1}(\tau) = \frac{\partial G}{\partial \varepsilon}(0, \tilde{f}_0(\varepsilon)) + \frac{\partial G}{\partial \tilde{f}}(0, \tilde{f}_0(\tau)) \tilde{f}_1(\tau),$$

and analogous equations for other $\tilde{f}_n(\tau)$.

If we choose the curves $\Gamma(\varepsilon)$ in the form

$$(3.13) \quad \tau_n = \varepsilon^n t, \quad n=0, 1, 2, \dots, N,$$

the Eqs. (3.11) and (3.12) become simpler

$$(3.14) \quad \frac{\partial \tilde{f}_0}{\partial \tau_0} = G(0, \tilde{f}_0(\tau)),$$

and

$$(3.15) \quad \frac{\partial \tilde{f}_1(\tau)}{\partial \tau_0} + \frac{\partial \tilde{f}_0(\tau)}{\partial \tau_1} = \frac{\partial G}{\partial \varepsilon}(0, \tilde{f}_0(\tau)) + \frac{\partial G}{\partial \tilde{f}}(0, \tilde{f}_0(\tau)) \tilde{f}_1(\tau).$$

The presence of the term $\partial \tilde{f}_0 / \partial \tau_1$ in the Eq. (3.15) illustrates the major differences between the extended perturbation theory (multiple time scale

method) and the conventional (direct) perturbation analysis. This term gives us the possibility to present the growth of $\tilde{f}$ for large τ_0 , and thus to assure the uniform (free of secular terms) validity of the expansion when the direct expansion of f shows a nonuniform („secular“) behaviour.

4. The Extension of Solutions of the Nonautonomous Differential Equation and the Construction of a „Superclock“. Consider a solution of the nonautonomous differential equation

$$(4.1) \quad \frac{df}{dt} = G(f, t)$$

with the initial condition

$$(4.2) \quad f(t_0) = a.$$

In order to obtain from the Eq. (4.1) an autonomous system, we consider a new function $h(t)$ as solution of the equation

$$(4.3) \quad \frac{dh}{dt} = 1, \quad h(t_0) = t_0.$$

Then the function (f, h) is the solution of the system

$$(4.4) \quad \begin{cases} \frac{df}{dt} = G(f, h), \\ \frac{dh}{dt} = 1, \end{cases}$$

with the conditions

$$(4.5) \quad f(t_0) = a, \quad h(t_0) = t_0.$$

We shall show how we can construct an extension $(\tilde{f}, \tilde{h} = T)$ of (f, h) along the curve (in the extended domain) given by the Eqs. (2.3)–(2.5). The function $\tilde{f}$ and T must satisfy the conditions

$$(4.6) \quad \tilde{f}(\tau_0(t), \dots, \tau_N(t)) = f(t)$$

and

$$(4.7) \quad T(\tau_0(t), \dots, \tau_N(t)) = h(t) \equiv t.$$

As in Sec. 2, $\tilde{f}$ and T verify the equations

$$(4.8) \quad \sum_{i=0}^N \left[\frac{\partial \tilde{f}(\tau)}{\partial \tau_i} g_i(\tau) \right]_{\Gamma} = G(\tilde{f}(\tau), T(\tau)),$$

$$(4.9) \quad \sum_{i=0}^N \left[\frac{\partial T(\tau)}{\partial \tau_i} g_i(\tau) \right]_{\Gamma} = 1.$$

These equations suggest the following procedure for the construction of the extension $(\tilde{f}, T)$. Let us consider the solution $(\tilde{f}, T)$ of the system

$$(4.10) \quad \sum_{i=0}^N \frac{\partial \tilde{f}(\tau)}{\partial \tau_i} g_i(\tau) = G(\tilde{f}, T),$$

$$(4.11) \quad \sum_{i=0}^N \frac{\partial T(\tau)}{\partial \tau_i} g_i(\tau) = 1,$$

with the initial conditions

$$(4.12) \quad f(a_0, a_1, \dots, a_N) = a,$$

$$(4.13) \quad T(a_0, a_1, \dots, a_N) = t_0.$$

As in the Sec. 2, it may be shown that, if we put $f_1(t) = f(\tau(t))$ and $h_1(t) = T(\tau(t))$, these functions are solutions of the Eqs. (4.4) and (4.5), that is, $f_1 \equiv f$ and $h_1 \equiv h$. Taking into account that $h(t) \equiv t$, we obtain the Eq. (4.6) and thus $\tilde{f}$ is an extension of f .

Let us consider a function T such that $T(\tau(t)) = t$. Then the function

$$(4.14) \quad \tilde{f} = f \circ T$$

is an extension of $f(t)$ along the trajectory Γ . This statement is called the induction theorem because it expresses the fact that an extension of a given function f may be represented with the function itself and an appropriate „supervariable“ T (see below).

As

$$(4.15) \quad \frac{df(T(\tau))}{dT} = G(f(T(\tau)), T(\tau)),$$

from the Eq. (4.11) we obtain

$$(4.16) \quad \frac{df}{dt} \sum_{i=0}^N \frac{\partial T}{\partial \tau_i} g_i = G(f \circ T, T),$$

or

$$(4.17) \quad \sum_{i=0}^N \frac{\partial (f \circ T)}{\partial \tau_i} g_i = G(f \circ T, T).$$

As expected (see the Eq. (4.12)), we find the obvious relation

$$(4.18) \quad \tilde{f}(a_0, a_1, \dots, a_N) = (f \circ T)(a_0, a_1, \dots, a_N) = a.$$

Thus $(\tilde{f}, T)$ is a solution of the system of equations (4.10)–(4.13) but it is not necessary that $\tilde{f} = f$.

The function $T(\tau) = T(\tau_0, \tau_1, \dots, \tau_N)$ is named „superclock“ or „super-variable“ since it summarizes the informations contained in the different (time, clock) scales τ_n ($n=0, 1, \dots, N$). We note that whenever we know a super-clock T along a trajectory Γ , we obtain $\tilde{f} = f \circ T$ as an extension of f .

5. Concluding Remarks. 1. In this work we present some procedures for construction of the extended functions in the cases of autonomous and

nonautonomous differential equations which may describe physical quantities with various on widely separated time scales.

2. These results can be extended to multiple space scales [8] when a disparity between space scales appear, as in the case of propagation of nonlinear waves in nonhomogeneous media.

Received July 27, 1990

Polytechnic Institute, Jassy,
Department of Physics
and
Department of Mathematics

REFERENCES

1. Davidson R. C., *Methods in Nonlinear Plasma Theory*. Academic Press, New York, 1972.
2. Montgomery D., Alexeff I., *Physics of Fluids*, **9**, 1362 (1968).
3. Choquet-Bruhat Y., *Commun. math. Phys.*, **12**, 16 (1969); Whitham G. B., *Studies in Applied Mathematics* (ed. A. H. Taub), Vol. 7, Mathematical Association of America Studies in Math., Englewood Cliffs, N. J., Prentice-Hall, 1971; Madore J., *Commun. math. Phys.*, **27**, 291 (1972).
4. Krylov N., Bogoliubov N., *Introduction to Non-Linear Mechanics*. Princeton Univ. Press, 1947.
5. Bogoliubov N., Mitropolskii I., *Les Méthodes Asymptotiques en Théorie des Oscillations non-linéaires*. Gauthier-Villars, Paris, 1962.
6. Frieman A. E., *J. Math. Phys.*, **4**, 410 (1963).
7. Sandri G., *Ann. Phys.*, **24**, 232 (1963); *Nuovo Cimento*, **36**, 67 (1965).
8. Jancel R., *Physica Scripta*, **13**, 373 (1976).

METODA SCALEI MULTIPLE-DE TIMP ȘI CONSTRUCȚIA UNUI „SUPERCEAS“

(Rezumat)

Se prezintă anumite proceduri pentru construcția funcțiilor extinse în cazul ecuațiilor diferențiale autonome și neautonome care pot descrie cantități fizice care variază în scale de timp complet diferite.

A MODEL FOR GRAVITATION ON THE $\mathbb{R} \times S^3$ TOPOLOGY

BY

GH. ZET, MARCEL AGOP, C. PASNICU, C. BRUMĂ, C. DĂRIESCU
 and MARINA DĂRIESCU

1. Introduction. The field theory on $\mathbb{R} \times S^3$ topology has been considered recently by different authors [1]... [3]. The reason is that the sphere S^3 is a space with considerable more symmetry than Minkowski space-time. It is convenient to have the temporal dimension continuous and flat, so the theory can be canonically quantized [4].

A model for gravitation on $\mathbb{R} \times S^3$ topology has been given by Carmeli and Malin [1]. In our work, we show that the curvature tensor given in [1] must be supplemented by a term due to the non-commutativity of the derivatives used on $\mathbb{R} \times S^3$ space.

2. The Gravitation on the $\mathbb{R} \times S^3$ Topology. On the $\mathbb{R} \times S^3$ space the standard partial differentiation operator is L_μ ($\mu=0, 1, 2, 3$) [3]

$$L_0 = \frac{\partial}{\partial t},$$

$$\begin{aligned} L_1 = & i \left(u \frac{\partial}{\partial v^*} - v \frac{\partial}{\partial u^*} + v^* \frac{\partial}{\partial u} - u^* \frac{\partial}{\partial v} \right) = \sin(\alpha + \beta) \frac{\partial}{\partial \theta} + \\ & + \cos(\alpha + \beta) \left(\tan \theta \frac{\partial}{\partial \alpha} - \cotan \theta \frac{\partial}{\partial \beta} \right), \\ (1) \quad L_2 = & \left(u \frac{\partial}{\partial v^*} - v \frac{\partial}{\partial u^*} - v^* \frac{\partial}{\partial u} + u^* \frac{\partial}{\partial v} \right) = \cos(\alpha + \beta) \frac{\partial}{\partial \theta} + \\ & + \sin(\alpha + \beta) \left(\tan \theta \frac{\partial}{\partial \alpha} - \cotan \theta \frac{\partial}{\partial \beta} \right), \\ L_3 = & i \left(u^* \frac{\partial}{\partial u^*} - u \frac{\partial}{\partial u} + v^* \frac{\partial}{\partial v^*} - v \frac{\partial}{\partial v} \right) = - \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \end{aligned}$$

with

$$\begin{aligned} (2) \quad u &= x_1 + ix_2 = \cos \theta e^{i\alpha}, \\ v &= x_3 + ix_4 = \sin \theta e^{i\beta} \end{aligned}$$

and they satisfy the commutation relations

$$(3) \quad [L_\mu, L_\nu] = 2\varepsilon_{0\mu\nu\sigma} L_\sigma,$$

where $\varepsilon_{0\mu\nu\sigma}$ is the Levi-Civita symbol ($\varepsilon_{0123}=1$).

Let $\{x^\mu=t, \theta, \alpha, \beta\}$ be a coordinate system in $\mathbb{R} \times S^3$. We introduce a connection in the tangent bundle $T(\mathbb{R} \times S^3)$ by defining the covariant derivatives in the $A=a^\nu L_\nu$ direction:

$$(4) \quad \nabla_A B = \nabla_{a^\nu L_\nu} (b^\mu L_\mu) = a^\nu [L_\nu(b^\mu) + b^\mu \Gamma_{\mu\nu}^\sigma L_\sigma],$$

where $\Gamma_{\mu\nu}^\sigma$ are the connection coefficients and they satisfy the relations

$$(5) \quad \nabla_{L_\nu} L_\mu = \Gamma_{\mu\nu}^\sigma L_\sigma.$$

The components of the curvature

$$(6) \quad R(L_\mu, L_\nu)L_\omega = R_{\omega\mu\nu}^\sigma L_\sigma$$

are obtained by the relations [3]

$$(7) \quad R(\nabla_\mu, \nabla_\nu) = [\nabla_\mu, \nabla_\nu] - \nabla_{[L_\mu, L_\nu]},$$

$$(8) \quad \nabla_{[L_\mu, L_\nu]} L_\omega = \nabla_{\varepsilon_{0\mu\nu\sigma} L_\sigma} L_\omega = \varepsilon_{0\alpha\nu}^\sigma \nabla_{L_\sigma} L_\omega = \varepsilon_{0\mu\nu}^\sigma \Gamma_{\alpha\sigma}^\rho L_\rho; \quad \varepsilon_{0\mu\nu}^\sigma \equiv \varepsilon_{0\mu\nu\sigma}$$

and (5); they have the following form in the L_μ base:

$$(9) \quad R_{\omega\mu\nu}^\sigma = L_\mu \Gamma_{\omega\nu}^\sigma - L_\nu \Gamma_{\omega\mu}^\sigma + \Gamma_{\omega\nu}^\rho \Gamma_{\rho\mu}^\sigma - \Gamma_{\omega\mu}^\rho \Gamma_{\rho\nu}^\sigma - 2\varepsilon_{0\mu\nu}^\rho \Gamma_{\omega\rho}^\sigma.$$

The last term on the right-hand side of Eq. (9) does not appear in the work [1] of Carmeli and Malin, and it is determined by the noncommutativity of the derivative L_μ . With this term, the $R_{\omega\mu\nu}^\sigma$ quantity is indeed a tensor.

The gravitational field equations are written under the form

$$(10) \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}; \quad k = \text{const.}$$

As usual, the energy-momentum tensor $T_{\mu\nu}$ will have specific forms according to the physics it describes.

Received February 2, 1990

Polytechnic Institute, Jassy,
Department of Physics

REFERENCES

1. Carmeli M., Malin S., Found. Phys., **17**, 407 (1987).
2. Sen D., Nucl. Phys., B **24**, 201 (1987).
3. Zet Gh., Found Phys., **20**, 1, 111 (1990).
4. Smolin L., Contemporary Mathematics, **71**, 55 (1988).

UN MODEL DE GRAVITAȚIE PE SFERA $\mathbb{R} \times S^3$

(Rezumat)

Se construiește teoria gravitației pe sfera $\mathbb{R} \times S^3$ prin definirea unei conexiuni pe fibratul tangent.

QUELQUES SIGNIFICATIONS PHYSIQUES SUR LA VARIÉTÉ DU GROUPE UNIMODULAIRE G_3

PAR

V. MELNIC, MARCEL AGOP, I. SANDU, MARIANA CAIA,
 MARINA DARIESCU et P. DURBACA

0. On trouve une signification quantique sur la variété du groupe unimodulaire G_3 .

1. Les transformations du groupe unimodulaire G_3 produites par des transformations infinitésimales. Le groupe unimodulaire Cartan a la forme [1]:

$$(1) \quad \begin{aligned} x' &= ax + b, & y' &= cx + d, \\ a, b, c, d &\in \mathbb{R}, & ad - bc &\equiv 1. \end{aligned}$$

Pour

$$(2) \quad a = 1 + k; \quad d = 1 - k; \quad k \in \mathbb{R}$$

et considérant b, c, k comme des grandeurs du première ordre, l'algèbre du groupe (1) est donnée par les opérateurs [2]

$$(3) \quad L_1 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad L_2 = y \frac{\partial}{\partial x}, \quad L_3 = x \frac{\partial}{\partial y}$$

et satisfait les relations de commutation

$$(4) \quad [L_1, L_2] = -2L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = -2L_2.$$

Les transformations finies générées par (3) sont déterminées comme des fonctions invariantes de l'opérateur [2]:

$$(5) \quad U = pL_1 + mL_2 + nL_3, \quad p, m, n = \text{const.},$$

c'est-à-dire de l'équation

$$(6) \quad U\psi = 0,$$

ou

$$(7) \quad (px + my) \frac{\partial \psi}{\partial x} + (nx - py) \frac{\partial \psi}{\partial y} \equiv 0.$$

Le système caractéristique associé au (7) est

$$(8) \quad \frac{dx}{px+my} = \frac{dy}{nx-py} = dt,$$

ce qui donne

$$(9) \quad \frac{\lambda dx + \mu dy}{\lambda(px+my) + \mu(nx-py)} = dt.$$

Il est nécessaire que (9) soit une différentielle totale exacte. Dans ces conditions le système

$$(10) \quad \begin{aligned} \lambda(p-\rho) + \mu n &= 0, \\ m\lambda - (p+\rho)\mu &= 0 \end{aligned}$$

doit être compatible. On trouve la relation

$$(11) \quad p^2 + mn = \rho^2,$$

où $\rho = (\pm 1; 0)$.

Pour $\rho = 1$ on obtient les transformations hyperboliques du groupe unimodulaire (1) [2]

$$(12) \quad \begin{aligned} x' &= (\operatorname{ch} t + p \operatorname{sh} t)x + m(\operatorname{sh} t)y, \\ y' &= n(\operatorname{sh} t)x + (\operatorname{ch} t - p \operatorname{sh} t)y. \end{aligned}$$

Si $\rho = -1$ [2]:

$$(13) \quad \begin{aligned} x' &= (\cos t + p \sin t)x + m(\sin t)y, \\ y' &= n(\sin t)x + (\cos t - p \sin t)y, \end{aligned}$$

c'est-à-dire les transformations unimodulaires elliptiques. Les transformations unimodulaires paraboliques

$$(14) \quad \begin{aligned} x' &= (pt+1)x + mty, \\ y' &= ntx + (1-pt)y \end{aligned}$$

s'obtiennent pour $\rho = 0$.

2. La signification quantique sur la variété du groupe unimodulaire G_3 . On sait que le groupe unimodulaire G_3 peut être organisé comme une variété différentiable; l'algèbre (3) du ce groupe est défini dans l'espace tangent de l'élément unité. Maintenant, les transformations hyperboliques (12) du groupe unimodulaire G_2 avec le choix

$$(15) \quad t = \gamma\tau, \quad p = \cos \Phi, \quad m = n = \sin \Phi$$

deviennent les transformations Stoler [3]

$$(16) \quad \begin{aligned} x' &= (\operatorname{ch} \gamma\tau + \operatorname{sh} \gamma\tau \cos \Phi)x + (\operatorname{sh} \gamma\tau \sin \Phi)y, \\ y' &= (\operatorname{sh} \gamma\tau \sin \Phi)x + (\operatorname{ch} \gamma\tau - \operatorname{sh} \gamma\tau \cos \Phi)y, \end{aligned}$$

c'est-à-dire la valeur propre de l'opérateur d'annihilation dans la deuxième quantification [4].

Si $|\psi_0\rangle$ est la fonction propre de l'opérateur d'annihilation avec la valeur propre (16), d'après [4] on obtient

$$(17) \quad \Delta p \Delta q = \frac{1}{2}(1 + \text{sh}^2 t m^2),$$

où Δq , Δp sont les indéterminations dans la coordonnée et l'impuls ($\hbar=1$). Cette relation montre que les états $|\psi_0\rangle$ ne sont pas des états de minime incertitude.

Reçue le 2 Mars, 1990

*Institut Polytechnique, Jassy,
Chaire de Physique
et
Chaire de Chimie Physique*

BIBLIOGRAPHIE

1. Cartan E., *Mémoires des Sc. Math.*, **42**, 21 (1930).
2. Vrinceanu Gh., *Lectii de geometrie diferențială*. Vol. III, Ed. did. și ped., Buc., 1979.
3. Stoler D., *Phys. Rev.*, **D1**, 3 217 (1970).
4. Carruthers P., Nieto M. M., *Rev. Mod. Phys.*, **40**, 411 (1968).

CÎTEVA SEMNIFICAȚII FIZICE PE VARIETATEA GRUPULUI UNIMODULAR G_3

(Rezumat)

Se găsește o semnificație cuantică pe varietatea grupului unimodular G_3 .

On June 1, 1900, the following property was donated to the
Department of the Interior, Bureau of Land Management, Washington, D.C.
by the State of California:

(1) 100 acres of land in the County of Santa Clara, State of California.

(2) 100 acres of land in the County of Santa Clara, State of California.
The above property was donated to the Department of the Interior, Bureau of Land Management, Washington, D.C. by the State of California.

The above property was donated to the Department of the Interior, Bureau of Land Management, Washington, D.C. by the State of California.

The above property was donated to the Department of the Interior, Bureau of Land Management, Washington, D.C. by the State of California.

The above property was donated to the Department of the Interior, Bureau of Land Management, Washington, D.C. by the State of California.

The above property was donated to the Department of the Interior, Bureau of Land Management, Washington, D.C. by the State of California.

The above property was donated to the Department of the Interior, Bureau of Land Management, Washington, D.C. by the State of California.

The above property was donated to the Department of the Interior, Bureau of Land Management, Washington, D.C. by the State of California.

NEW ASPECTS IN THE INVARIANTIVE MECHANICS

BY

I. SANDU, MARCEL AGOP, GH. ZET, V. MELNIC, NICULINA DURBACA,
 MARIANA CAIA, C. DĂRIESCU and VIOLETA GHIURCAN

1. Introduction. The de Broglie's theory has been considered recently in [1]. The reason is that the clock's synchronisation in the invariantive mechanics is given by the Barbilian's symmetry. Because this theory must be canonically quantized [3], it requires the $\mathbb{R} \times H^3$ topology, with $\mathbb{R}$ the temporal dimension continuous and flat, H^3 the manifold of the Barbilian's groupe.

In this paper we give in the invariantive mechanics a model for gravitation on $\mathbb{R} \times H^3$ topology like a deformation theory of the physical frame.

2. The Geometry of $\mathbb{R} \times H^3$ Space. The Barbilian's groupe (H) is given by the homographical transformations [1]

$$(1) \quad h' = \frac{ah+b}{ch+d}; \quad k' = \frac{\bar{c}\bar{h}+d}{ch+d} k,$$

with

$$(2) \quad h = u + iv, \quad \bar{h} = \bar{u} - i\bar{v}, \quad k = e^{i\psi}.$$

This group has the 1-forms absolutely invariant

$$(3) \quad \omega_0 = i \left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h - \bar{h}} \right); \quad \omega_1 = \bar{\omega}_2 = \frac{dh}{k(h - \bar{h})}$$

and the 2-forms

$$(4) \quad \frac{ds^2}{g} = (\omega_0^2 - 4\omega_1\omega_2) = - \left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h - \bar{h}} \right)^2 + 4 \frac{dh d\bar{h}}{(h - \bar{h})^2};$$

$g = \text{const.}$

(4) is the Cayleyan metric of the projective space taking as absolute a one sheet hyperboloid [1].

We choose in this space (H^3) the differentiation operators P_i ($i=1, 2, 3$) [1]

$$(5) \quad \begin{aligned} P_1 &= \frac{\partial}{\partial h} + \frac{\partial}{\partial \bar{h}}, & P_2 &= -h \frac{\partial}{\partial h} - \bar{h} \frac{\partial}{\partial \bar{h}}, \\ P_3 &= h^2 \frac{\partial}{\partial h} + \bar{h}^2 \frac{\partial}{\partial \bar{h}} - (h - \bar{h})k \frac{\partial}{\partial k} \end{aligned}$$

and, in the four-dimensional space $(\mathbb{R} \times H^3)$, the operators $P_\mu \equiv (P_\mu, \partial/\partial t)$. Note that these derivatives do not commute

$$(6) \quad [P_\mu, P_\sigma] = f_{\mu\sigma}^\omega P_\omega,$$

namely

$$(7) \quad f_{12}^1 = f_{23}^3 = -1; \quad f_{31}^2 = -2; \quad f_{i4}^i = 0; \quad i = 1, 2, 3.$$

Here, $f_{\mu\sigma}^\omega = -f_{\sigma\mu}^\omega$ are the structure constants of the $\mathbb{R} \times H$ group where $\mathbb{R}$ is the translation group in time and H is the Barbilian's group. Consequently, $\partial/\partial t$ is the $\mathbb{R}$ -generator and P_i the H -generators.

3. The Gravitation on the $\mathbb{R} \times H^3$ Topology. Let $(x^\mu = h, \bar{h}, k, t)$ be a coordinate system in $\mathbb{R} \times H^3$. We introduce a connection in the tangent bundle $T(\mathbb{R} \times H^3)$ by defining the covariant derivatives in the $A = a^\nu P_\nu$ direction [4]

$$(8) \quad \nabla_A B = \nabla_{a^\nu P_\nu} (b^\mu P_\mu) = a^\nu [P_\nu (b^\mu) + b^\mu \Gamma_{\mu\nu}^\sigma] P_\sigma,$$

where $\Gamma_{\mu\nu}^\sigma$ are the connection coefficients and they satisfy the relations

$$(9) \quad \nabla_{P_\nu} P_\mu = \Gamma_{\mu\nu}^\sigma P_\sigma.$$

The components of the curvature

$$(10) \quad R(P_\mu, P_\nu) P_\omega = R_{\omega\mu\nu}^\sigma P_\sigma$$

are given by the following relation [4]:

$$(11) \quad R_{\omega\mu\nu}^\sigma = P_\mu \Gamma_{\omega\nu}^\sigma - P_\nu \Gamma_{\omega\mu}^\sigma + \Gamma_{\omega\nu}^\rho \Gamma_{\rho\mu}^\sigma - \Gamma_{\omega\mu}^\rho \Gamma_{\rho\nu}^\sigma - f_{\mu\nu}^\rho \Gamma_{\rho\omega}^\sigma.$$

The last term on the right-hand side of Eq. (9) do not appear in the work [2] of Carmeli and Malin, being determined by the noncommutativity of the derivative P_μ . With this term, the $R_{\omega\mu\nu}^\sigma$ quantity is indeed a tensor.

The torsion of the connection (8) is nought. Indeed, the formula [5]

$$(12) \quad T(A, B) = \nabla_A B - \nabla_B A - [A, B],$$

with (9) gives

$$(13) \quad T(P_\mu, P_\nu) P_\omega = \{\nabla_{P_\mu} P_\nu - \nabla_{P_\nu} P_\mu - [P_\mu, P_\nu]\} P_\omega = T_{\mu\nu}^\omega P_\omega,$$

or, using relation (9) and the property of the structure constants

$$(14) \quad T_{\mu\nu}^\omega P_\omega = (\Gamma_{\nu\mu}^\omega - \Gamma_{\mu\nu}^\omega - f_{\mu\nu}^\omega) P_\omega.$$

In these conditions, the torsion tensor have the expression

$$(15) \quad T_{\mu\nu}^\omega = \Gamma_{\nu\mu}^\omega - \Gamma_{\mu\nu}^\omega - f_{\mu\nu}^\omega.$$

But [5]

$$(16) \quad \nabla_{P_\mu} P_\nu - \nabla_{P_\nu} P_\mu = (\Gamma_{\nu\mu}^\omega - \Gamma_{\mu\nu}^\omega) P_\omega = (P_\mu P_\nu - P_\nu P_\mu) = [P_\mu, P_\nu] = f_{\mu\nu}^\omega P_\omega,$$

that is

$$(17) \quad \Gamma_{\nu\mu}^\omega - \Gamma_{\mu\nu}^\omega - f_{\mu\nu}^\omega = 0.$$

Thus, the torsion tensor is

$$T_{\mu\nu}^{\omega}=0.$$

4. The Deformation of Physical Frame and the Gravitation. It is known that the physical frame in the invariantive mechanics is given by the coordinate system $(h, \bar{h}, k, t)$. But, the same coordinate system is involved in the deformation theory [1] for anisotropic materials. In this conditions, if we consider the Einstein equations

$$(19) \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}; \quad \kappa = \text{const.}$$

like a constitutive relation between the space and the matter, then the gravitation is given by the deformation of the physical frame.

Received February 2, 1990

Polytechnic Institute, Jassy,
Department of Physics

REFERENCES

1. Agop M., Mazilu N., *Fundamentele fizicii moderne*. Ed. „Junimea”, Iași, 1989.
2. Carmeli M., Malin S., *Found. Phys.*, **17**, 407 (1987).
3. Smolin L., *Contemporary Mathematics*, **71**, 55 (1988).
4. Zet Gh. et al., *Bul. Inst. Polit. Iași* (submitted).
5. Gheorghiev Gh., Oproiu V., *Varietăți diferențiabile finite și infinite dimensionale*. Vol. I, Ed. Acad. Române, București, 1976.

NOI ASPECTE ÎN MECANICA INVARIANTIVĂ

(Rezumat)

Se arată că în mecanica invariantivă deformarea reperului fizic poate fi echivalentă cu gravitația.

LES ÉQUATIONS LAGRANGE, HAMILTON ET HAMILTON-JACOBI SUR LA VARIÉTÉ $E^3 \times T$

PAR

MARCEL AGOP, C. DĂRIESCU et MARINA DĂRIESCU

1. Introduction. En utilisant le formalisme de la mécanique invariante on retrouve des résultats de la mécanique analytique.

Dans ce but en acceptant la 1-forme [1]

$$(1) \quad \omega_\delta = p_k \delta q^k - H \delta t = L \delta t; \quad k=1, 2, \dots, n,$$

où les grandeurs physiques ont les significations connues [2], et la différentielle extérieure

$$(2) \quad \begin{aligned} D\omega_\delta = d\omega_\delta - \delta\omega_\delta = dp_k \delta q^k - \delta p_k dq^k + \\ + p_k (d\delta - \delta d) q^k - dH \delta t + \delta H dt - H(d\delta - \delta d)t, \end{aligned}$$

ou

$$(3) \quad D\omega_\delta = dL \delta t - \delta L dt + L(d\delta - \delta d)t,$$

on écrit que la mécanique invariante introduit le principe du mouvement

$$(4) \quad D\omega_\delta = 0.$$

Pour la variété $E^3 \times T$, l'espace de la mécanique invariante

$$(5) \quad q \rightarrow r, \quad \omega_\delta \rightarrow p \delta r - H \delta t,$$

$$(6) \quad (d\delta - \delta d) = [d, \delta] = 0,$$

et (2), (3) donnent

$$(7) \quad D\omega_\delta = dp \delta r - \delta p dr - dH \delta t + \delta H dt,$$

respectivement

$$(8) \quad D\omega_\delta = dL \delta t - \delta L dt.$$

Le principe (4) permet d'obtenir des résultats bien connus dans la mécanique analytique.

2. L'équation Hamilton-Jacobi. La 1-forme (5) multipliée avec $(dt)^{-1}$ donne

$$(9) \quad \frac{dS}{dt} = \mathbf{p} \cdot \mathbf{v} - H, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

Mais, on sait que [2]

$$(10) \quad \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \mathbf{r}} \mathbf{v} = \frac{\partial S}{\partial t} + \mathbf{p} \cdot \mathbf{v},$$

où

$$(11) \quad \frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}.$$

Dans ces conditions (9) prend la forme connue [2]

$$(12) \quad \frac{\partial S}{\partial t} + H = 0.$$

3. Les équations de Hamilton. Parce que

$$(13) \quad \delta H = \frac{\partial H}{\partial t} \delta t + \frac{\partial H}{\partial \mathbf{r}} \delta \mathbf{r} + \frac{\partial H}{\partial \mathbf{p}} \delta \mathbf{p},$$

la différentielle extérieure (7) devient

$$(14) \quad \mathcal{D}\omega_\delta = \left(d\mathbf{p} + \frac{\partial H}{\partial \mathbf{r}} \right) \delta \mathbf{r} + \left(-d\mathbf{r} + \frac{\partial H}{\partial \mathbf{p}} \right) \delta \mathbf{p} - dH \delta t + \frac{\partial H}{\partial t} \delta t;$$

comme les variations $\delta \mathbf{r}$, $\delta \mathbf{p}$, δt sont arbitraires, leurs coefficients doivent être nuls. Par la multiplication avec $(dt)^{-1}$ la relation (14) donne :

a) les équations de Hamilton

$$(15) \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}}, \quad \frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}};$$

b) l'équation de la variation de l'énergie

$$(16) \quad \frac{dH}{dt} = -\frac{\partial H}{\partial t}.$$

Dans le cas tout particulier des systèmes holonomes (16) se réduit à l'équation de l'énergie

$$(17) \quad \frac{dH}{dt} = 0.$$

4. Les équations de Lagrange. Ayant en vue que

$$(18) \quad \delta L = \frac{\partial L}{\partial \mathbf{r}} \delta \mathbf{r} + \frac{\partial L}{\partial \dot{\mathbf{r}}} \delta \dot{\mathbf{r}} + \frac{\partial L}{\partial t} \delta t,$$

$$(19) \quad \frac{\partial L}{\partial \dot{\mathbf{r}}} \delta \dot{\mathbf{r}} = \frac{\partial L}{\partial \dot{\mathbf{r}}} \frac{d}{dt} (\delta \mathbf{r}) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \delta \mathbf{r} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \right) \delta \mathbf{r},$$

la différentielle extérieure (8), multipliée avec $(dt)^{-1}$ devient

$$(20) \quad \mathcal{D}\omega_s = \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \right) - \frac{\partial L}{\partial \mathbf{r}} \right] \delta \mathbf{r} + \left(\frac{dL}{dt} - \frac{\partial L}{\partial t} \right) \delta t - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \delta \mathbf{r} \right) = 0.$$

Comme la variation $\delta \mathbf{r}$ est arbitraire, la relation (19) donne les équations de Lagrange

$$(21) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \right) - \frac{\partial L}{\partial \mathbf{r}} = 0.$$

La relation

$$(22) \quad \left(\frac{dL}{dt} - \frac{\partial L}{\partial t} \right) \delta t = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \delta \mathbf{r} \right),$$

multipliée par $(\delta t)^{-1}$ et en tenant compte que $\partial \mathbf{r} / \partial t = \mathbf{v}$, donne l'équation de la variation de l'énergie

$$(23) \quad \frac{dH}{dt} = - \frac{\partial L}{\partial t}.$$

Pour les systèmes holonomes (22) se réduit à (17).

5. Conclusions. Le formalisme de la mécanique invariante permet d'obtenir les équations d'évolution des systèmes physiques dans une manière plus générale que celle de la mécanique analytique, parce que la condition $\delta t \neq 0$ conduit, en vertu du principe de Heisenberg [3], à l'existence des familles de trajectoires autour de la trajectoire réelle (classique).

On peut l'interpréter comme un passage vers la théorie moderne du "path integrals" [4].

Reçue le 2 Février, 1990

Institut Polytechnique, Jassy,
Chaire de Physique

BIBLIOGRAPHIE

1. Onicescu O., *Invariantive Mechanics*. Springer-Verlag, Wien, 1975.
2. Plăcișteanu I., *Mecanică Analitică și Vectorială*. Ed. tehnică, București, 1958.
3. Landau L., Lifschitz E., *Mécanique Quantique*. Éd. Mir, Moscou, 1980.
4. Chaichian M., Nalipa N. F., *Introduction to Gauge Field Theories*. Springer Verlag, Berlin—Heidelberg—New York—Tokio, 1988.

ECUAȚIILE LAGRANGE, HAMILTON ȘI HAMILTON-JACOBI PE VARIETATEA $E^3 \times T$

(Rezumat)

Se obțin ecuațiile generale de evoluție a sistemelor fizice clasice folosind un principiu de mișcare mai general decât cel de staționaritate.

(1) The first part of the document is a letter from the President of the United States to the Congress, dated January 1, 1863. It is a very important document, as it contains the President's message to Congress, which is a key part of the executive branch's communication with the legislative branch.

(2) The second part of the document is a letter from the Secretary of the Treasury to the President, dated January 1, 1863. It is a very important document, as it contains the Secretary's report to the President, which is a key part of the executive branch's communication with the legislative branch.

(3) The third part of the document is a letter from the Secretary of the Treasury to the President, dated January 1, 1863. It is a very important document, as it contains the Secretary's report to the President, which is a key part of the executive branch's communication with the legislative branch.

(4) The fourth part of the document is a letter from the Secretary of the Treasury to the President, dated January 1, 1863. It is a very important document, as it contains the Secretary's report to the President, which is a key part of the executive branch's communication with the legislative branch.

(5) The fifth part of the document is a letter from the Secretary of the Treasury to the President, dated January 1, 1863. It is a very important document, as it contains the Secretary's report to the President, which is a key part of the executive branch's communication with the legislative branch.

(6) The sixth part of the document is a letter from the Secretary of the Treasury to the President, dated January 1, 1863. It is a very important document, as it contains the Secretary's report to the President, which is a key part of the executive branch's communication with the legislative branch.

(7) The seventh part of the document is a letter from the Secretary of the Treasury to the President, dated January 1, 1863. It is a very important document, as it contains the Secretary's report to the President, which is a key part of the executive branch's communication with the legislative branch.

ON THE ULTRASONIC PROPAGATION IN FERROFLUIDS

BY

EMIL LUCA and R. BĂDESCU

An explanation is suggested of some aspects connected to the ultrasonic propagation in ferrofluids. It is shown that, in the presence of a magnetic field, an additional absorption of ultrasonics occurs in ferrofluids.

A ferrofluid consists of a suspension of ferromagnetic particles in a basic fluid [1]. The suspension is obtained by means of a stability agent. Given its ability to get magnetized to saturation at relatively small magnetic field intensities ($5 \cdot 10^5$ A/m), the ferrofluids have large applications: for sealings [2], as gilling material for loudspeaker magnetic heads [3], in controlled acoustic contacts, as vibration transducers a. s. o. That is why the ferrofluid behaviour in magnetic field and the influence on the ultrasonic propagation in ferrofluid present a special interest.

In most cases the ferrofluids are considered hydrodynamic continuous media. Yet, it is well known that there is a series of phenomena produced only by a discrete structure of them [4]. The discrete structure is given by magnetic particles in suspension and especially by the agglomerations produced in ferrofluids situated in magnetic fields.

A liquid magnetizable medium can induce modifications in the parameters of the ultrasonic propagating through it and vice-versa, an ultrasonics can produce modifications in the ferrofluid parameters.

An explanation of the phenomenon has been tried in ultrasonic propagation in ferrofluids in the presence of a magnetic field, considering the particles in suspension as spheres in an isotropic elastic matrix [5]. This approximation is not very exact since the microstructure associated to ferrofluids are not ideal ones, but they contain particles of irregular shape and elastically anisotropic.

For the study of wave dispersion by the particles in suspensions, the following three cases are presented according to the ratio between the wave length and the particle diameter: $\lambda/D \ll 1$ when the dispersion is diffused; $\lambda/D \approx 1$ when a stochastic dispersion is considered and $\lambda/D \gg 1$ when can be extended the Rayleigh analysis of the wave power scattered by an elastic particle situated in a fluid.

These three theories indicate an increase of the attenuation coefficient when the frequency increases; it was theoretically demonstrated that this increase is proportional to f^4 , but the experiment showed to be proportional

to f^2 . Hence the analysis suggested for the monophase systems is no longer valid for the multi-phase systems.

The relation suggested for the attenuation coefficient and scattering coefficient of the ultrasonic waves in ferrofluids has the form [6]

$$(1) \quad 2\alpha = \varepsilon \frac{k^4 a^3 / 3 + k(\sigma - 1)^2 S}{S^2 + (\sigma + \tau)^2},$$

where α is the damping coefficient, ε — the volumic concentration of the particles in suspension, $k = 2\pi/\lambda$ is the wave-vector, a — the particles radius, $\sigma = \rho_1/\rho_0$ with ρ_1 the particles density and ρ_0 density of base liquid,

$$(2) \quad S = \frac{9}{4\beta a} \left(1 + \frac{1}{\beta a} \right), \quad \tau = \frac{1}{2} + \frac{9}{4\beta a},$$

$$(3) \quad \beta = \left(\frac{\omega}{2\mu} \right)^{1/2}, \quad \text{with } \omega = 2\pi f.$$

The relation (1) is no longer valid for too high concentrations and does not explain the dependence of the attenuation factor on the suspended particles diameter and the magnetic field.

Recent researches [7]... [9] lead to a new expression for ultrasonic attenuation coefficient in ferrofluids

$$(4) \quad \alpha = \frac{1}{2} m^2 X \frac{\omega}{a_0} \left(\sin^2 \gamma + \frac{\cos^2 \gamma}{k} \right),$$

where γ is the angle between the wave vector and the magnetic field; $k = 2\pi/\lambda$ is the wave vector having a component parallel to the magnetic field ($k_{||}$) and a component perpendicular to the magnetic field ($k_{\perp}$), these components of the wave vector depending on the density of the ferrofluid, on its viscosity and the degree of particles agglomeration; c_0 is the initial speed of ultrasonics; $X = X(d; d/\lambda) = \varphi \tau / k_{\perp}$, where $\omega = 2\pi f$ and τ is the relaxation time of the ferrofluid magnetization, that depends on density, viscosity and particles agglomeration degree in ferrofluid.

From the expression of the ultrasonic attenuation coefficient one can see that it depends on the direction of the magnetic field. The theoretically dependence of the ultrasonic attenuation factor on the direction of magnetic field is given in Fig. 1.

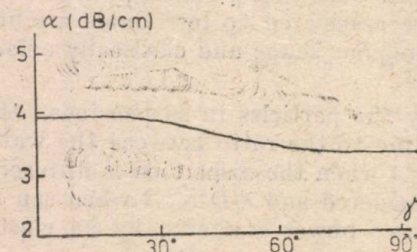


Fig. 1

The experimental determination of the relaxation time for a constant value of the magnetic field have shown that the ultrasonic attenuation coefficient changes periodically with the angle between the direction of the magnetic field and ultrasonic wave propagation direction. This can be explained as follows: when the

angle between the magnetic field direction and wave propagation direction is changing, the dimensions of the particles agglomerations are also chan-

ging. The wave scattering on this agglomeration is different in different directions.

The new aspects vendeder evident by the ultrasonic propagation in ferrofluids impose the acceptance of the idea of a complex structure of the ferrofluid, different from the simple model suggested by *Rosensweig*.

Received December 12, 1990

*Polytechnic Institute, Jassy,
Department of Physics*

REFERENCES

1. *Rosensweig R. E., Neuringer J. L., The Physics of Fluids, 7, 12, 1927 (1964).*
2. Библик Е. Е., Колоидный Ж., **35**, 6, 1141 (1973).
3. Библик Е. Е., Колоидный Ж., **36**, 3, 324 (1974).
4. Сапошников Г. И., Слиомис И. М., Магнитная гидродинамика, **2**, 47 (1975).
5. Luca E., Bădescu R., Bădescu V., Lucrările celei de-a II-a Conferințe Naționale de Magnetism, Iași, 1980.
6. Gotoh K., Chung D., J. Phys. Soc. Japan, **53**, 8, 2 521 (1984).
7. Полунин В. М., Акустический Ж., **21**, 2, 234 (1985).
8. Гогосов В. В., Магнитная гидродинамика, **2**, 19 (1987).
9. Гогосов В. В., Магнитная гидродинамика, **3**, 15 (1987).
10. Сурков В. В., Магнитная гидродинамика, **2**, 9 (1989).

ASUPRA PROPAGĂRII ULTRASUNETELOR ÎN FEROFUIDE

(Rezumat)

Se propun explicații ale unor aspecte privind propagarea ultrasunetelor în ferrofluide. Se arată că în prezența unui câmp magnetic, în ferrofluide apare o absorbție suplimentară a ultrasunetelor. Efectul este pus pe seama structurii discrete a ferrofluidelor și a aglomerărilor care apar în acestea în prezența câmpului magnetic.

The first thing I noticed when I stepped out of the car was the cold. It was a sharp contrast to the warm blanket of the car. I looked around, trying to get my bearings. The street was empty, and the buildings were old and weathered. I felt a sense of isolation, but also a sense of freedom. I had been told that this was a good place to live, and now I was here. I took a deep breath and walked towards the building.

I had been told that this was a good place to live, and now I was here. I took a deep breath and walked towards the building. The door was open, and I stepped inside. The interior was dimly lit, and the air was thick with the smell of old wood. I looked around, trying to get my bearings. The room was large and open, with a high ceiling and a large window. I felt a sense of peace, and I knew that this was my home.

I had been told that this was a good place to live, and now I was here. I took a deep breath and walked towards the building. The door was open, and I stepped inside. The interior was dimly lit, and the air was thick with the smell of old wood. I looked around, trying to get my bearings. The room was large and open, with a high ceiling and a large window. I felt a sense of peace, and I knew that this was my home.

THE CORRELATION BETWEEN THE POSITION OF THE TSD PEAKS AND THE FORMATION AND STORAGE CONDITIONS

BY

EUGEN NEAGU, RODICA NEAGU and CLAUDIA BOTEZ

1. Introduction. The essence of the thermally stimulated discharge experiment (TDS) is the formation of the polymer electret. This involves placing a thin layer of a polymer sample in a dielectric cell, raising the sample temperature to a level in excess to T_g and applying a d. c. voltage for a predetermined time t_f . The d. c. voltage generates a continuum electric field E . So, the forming parameters of polymer electret are: the electric field E the temperature T_f and the time t_f (Fig. 1). This causes a degree of alignment,

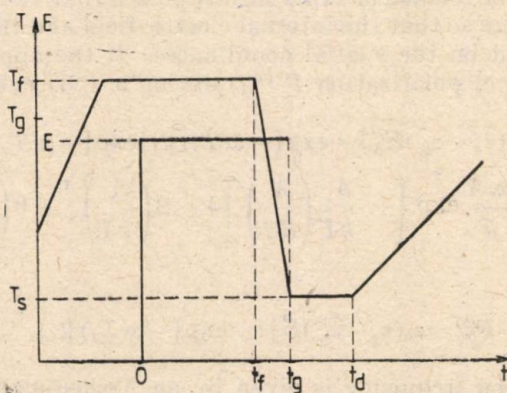


Fig. 1

of the dipoles and a drift of free charges to the electrodes. While maintaining the field the sample is cooled to below T_g , during $t_r = t_s - t_f$ to freeze in molecular motions. The electric field is there removed and the sample short-circuited during $t_c = t_d - t_s$ across an electrometer, which is able to monitoring low currents. After these operations which lead us to the achievement of the polymer electret, the cell is heated at a constant rate $b = dT/dt$ ($s = b^{-1}$) and different molecular modes become activated at specific temperatures. This motion of the dipolar groups results in image charges being released and this is measured as the TSC or TSD current peak on the thermogram. The main variable are: the electret forming conditions mentioned above and the TSD scan rate b .

In this paper one presents the influence of the formation and storage parameters of the electrets in the expressions of polarization, current density and temperature where the peak occurs in an TSD experiment. One compares the expressions of these quantities resulting from linear theory accepted nowadays and the nonlinear theory proposed by the authors. One discusses some consequences of the nonlinear theory.

2. Theory of Thermally Stimulated Current Measurement. In the following paragraph we shall briefly present the linear theory accepted nowadays with the help of which one can explain the TSD peaks by dipolar orientation and one can deduce the characteristic parameters for the studied dielectrics (particularly the polymers which contain polar side-chains). The dipolar relaxation times can be unique or distributed according to certain distribution functions. The TSD current theory produced by a unique time of dipolar relaxation was given firstly by C. Bucci, R. Fieschi and G. Guidi [1]. This theory was generalized by B. Gross [2] and J. van Turnhout and latter completed by the supposition that the dipolar relaxation times or the activation energies are distributed, using the distribution functions of Gevers, Gross, Wagner, Cole-Cole and Fuoss-Kirkwood [3], [4].

First, one presents the main equations deduced using the linear theory for the thermostimulated current produced by dipolar relaxation characterized by a unique time of relaxation. One also supposes that the polymer does not contains free charges so that the internal electric field and the dipolar polarization do not depend on the spatial coordinates. If the applied field is weak, the time variation of polarization $P^{(1)}(t)$ during a TSD cycle is given by

$$(1) \quad P^{(1)}(t) = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)E \{1 - \exp[-\alpha(T_f)t_f]\} \exp[-\alpha(T_s)(t_d - t_s)] \cdot \\ \cdot \exp\left\{-\frac{s\alpha_r A}{k} \exp\left\{-\frac{A}{kT} \left(\frac{A}{RT}\right)^2 \left[1 - 2\left(\frac{A}{kT}\right)^{-1} + 6\left(\frac{A}{kT}\right)^{-2}\right]\right\}\right\},$$

with

$$P_0^{(1)} = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)E \{1 - \exp[-\alpha(T_f)t_f]\},$$

where the relaxation frequency is given by an Arrhenius type relaxation

$$(2) \quad \alpha(T) = \alpha_r e^{-\frac{A}{kT}}, \quad \tau = \tau_0 e^{\frac{A}{kT}},$$

A is the activation energy and τ is the relaxation time. The depolarization current density is the time derivative of this expression. The peak of the current density in the linear theory appears at the temperature

$$(3) \quad T_m = \left(\frac{A}{\alpha(T_m)sk}\right)^{\frac{1}{2}} = \left(\frac{bA\tau(T_m)}{k}\right)^{\frac{1}{2}}.$$

In the nonlinear theory proposed by us [5], [6] the time variation of the polarization $P^2(t)$ is given by

$$(4) \quad P^2(t) = P_0^{(2)} \exp \left[-M - \frac{s\alpha_r A}{K} \exp(-Z) Z^{-2} (1 - 2Z^{-1} + 6Z^{-2}) \right] \cdot \\ \cdot \left[1 + \frac{P_0^{(2)} \mu}{3\varepsilon_0} \frac{s\alpha_r \exp(-M)}{k} \exp(-Z) Z^{-1} (1 - Z^{-1} + 2Z^{-2}) \right],$$

where

$$Z = \frac{A}{kT}, \quad M = \alpha(T_s)(t_a - t_s),$$

$$(5) \quad P_0^{(2)} = \varepsilon_0(\varepsilon_s - \varepsilon_\infty) E \left\{ 1 - \left[\frac{3kT_f}{\mu E} - 3 - (\varepsilon_s - \varepsilon_\infty) \right] \cdot \right. \\ \cdot \left. \frac{\mu E/kT_f}{1 - (\mu E/kT_f)[1 + \frac{1}{3}(\varepsilon_s - \varepsilon_\infty)]} \exp \left\{ -\alpha(T_f)t_f \left[1 - \frac{\mu E}{kT_f} \left(1 + 2\frac{\varepsilon_s - \varepsilon_\infty}{3} \right) \right] \right\} \right\}.$$

The peak of the current density results from

$$(6) \quad \frac{A}{kT_m s} = \alpha(T_m) - \frac{\mu(P^{(2)}(T_m))}{3\varepsilon_0 k T_m^2 s} \left[1 - \frac{\mu P^{(2)}(T_m)}{3\varepsilon_0 k T_m} \right]^{-1}.$$

This expression shows that the position at which the peak appears depends through the expression of $P^{(2)}(T_{\max})$ on the forming and storage conditions of the thermoelectret. If in (1) and (4) one notes

$$(7) \quad \frac{T}{T_m} = x, \quad y_1(x) = \frac{P^{(1)}(T)}{P_0^{(1)}}, \quad y_2(x) = \frac{P^{(2)}(T)}{P_0^{(2)}}, \\ a = s\alpha_r \frac{A}{k} = \left(\frac{A}{kT_m} \right)^2 \exp \left(\frac{A}{kT_m} \right), \\ b = \frac{P_0^{(2)} \mu}{3\varepsilon_0 k} s\alpha_r = \frac{P_0^{(2)} \mu}{3\varepsilon_0 k} \left(\frac{A^{\frac{1}{2}}}{k^{\frac{1}{2}} T_m} \right)^2 \exp \left(\frac{A}{kT_m} \right),$$

one gets for y_1 and y_2 the relations:

$$(8) \quad y_1(x) = \exp \left[-a \exp \left(\frac{A}{kT_m} \right) \frac{1}{x} \right] \left(\frac{kT_m}{A} \right)^2 x^2 \left[1 - 2 \frac{kT_m}{A} x + 6 \left(\frac{kT_m}{A} \right)^2 x^2 \right], \\ y_2(x) = y_1(x) + y_1(x)b \exp \left(-\frac{A}{kT_m} \frac{1}{x} \right) \frac{kT_m}{A} x \left[1 - \frac{kT_m}{A} x + 2 \left(\frac{kT_m}{A} \right)^2 x^2 \right], \\ D(x) = y_2(x) - y_1(x).$$

The domain of variability of $y_1(x)$ and $y_2(x)$ is (0, 1) and the domain of variability of x is considered (0.8; 1.2). The values of the involved parameters are given in Table 1.

Table 1

μ cm	E V/m	$\frac{A}{kT_m}$	$\epsilon_1 - \epsilon_\infty$	a	b
10^{-20}	10^6	$\frac{100}{3}$	2	$\left(\frac{100}{3}\right)^2 \frac{100}{e^3}$	$4.14 \frac{100}{66e} \frac{100}{e^3}$

Taking into account the expressions (8), we have calculated the values of $y_1(x)$ and $y_2(x)$ in the domain of variability considered for x . In Fig. 2 we have represented the function $D(x)$ for $A/kT_m = 100/3$ and the function $y_1(x)$ for the same value of A/kT_m .

3. Discussion. 1. From expressions (8) and Fig. 2 we conclude that the corrections which result if we take into account the nonlinear theory are im-

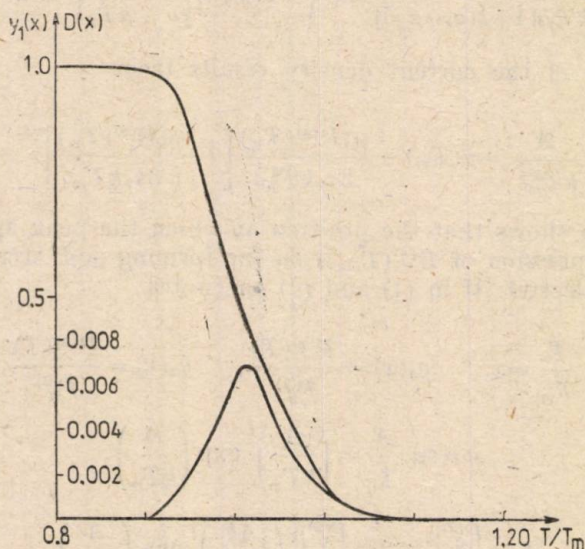


Fig. 2

portant if the parameter A/kT_m has small values. It also results that in the case of the linear theory, T_m given by expression (3) does not depend on the forming and storage conditions, but only on the scan rate of temperature b . The greater the heating rate is, the greater is the shift of the temperature towards higher values.

2. If the heating rate is constant, T_m should remain unchanged if the values of forming electric field E change. This is not usually noticed experimentally. The shift of the temperature T_m towards higher values, with the increase of the forming electric field E , as observed experimentally [7], can be attributed to nonlinear effects.

Received October 17, 1990

Polytechnic Institute, Jassy,
Department of Physics

REFERENCES

1. Bucci C., Fieshi R., Guidi G., Phys. Rev., 198 (1966).
2. Gross B., Appl. Optics Suppl., 3 NY (1969).
3. van Turnhout J., *Thermally Stimulated Discharge of Polymer Electrets*, Elsevier, 1977.
4. Sessler G. M. (Ed.), *Electrets*, Springer Verlag, 1980.
5. Neagu R., Bul. Inst. Polit. Iași, XXVII(XXXI), 1-2, s. I (1981).
6. Neagu R., Rev. Roum. de Phys., 26, 729 (1981).
7. Neagu E., Negulescu I., Neagu R., *The Thermally Stimulated Current Technique Applied to poly(γ -benzil-L-glutamat) Liquid-Crystal Polymers* (to be published).

CORELAȚIA ÎNTRE POZIȚIA MAXIMELOR TSD ȘI CONDIȚIILE DE FORMARE ȘI STOCARE

(Rezumat)

Se prezintă influența parametrilor de formare și stocare a electreților asupra expresiilor polarizației densității de curent relaxat și a temperaturii la care apare maximul curentului relaxat, într-un experiment TSD. Se compară expresiile acestor mărimi care rezultă din teoria liniară acceptată azi și teoria neliniară propusă de autori. Se discută unele consecințe care decurg din teoria neliniară.

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE UNIVERSITY OF CHICAGO PRESS

THE DRAG PHENOMENON AND THE HALL EFFECT IN SEMICONDUCTORS WITH NONUNIFORM BANDS

BY

MARIANA ISTRATE and SERGIU ISTRATE

1. Introduction. In a semiconductor with a position-dependent band structure, the bottom of the conduction band E_c , the top of the valence band E_v , the band gap E_g , the density of states g as well as the electron affinity χ are dependent of coordinates. These semiconductors present a graded composition of two or more components, a dependence on position of the impurity doping, a nonuniform strain or a nonuniform temperature.

J. Bardeen and W. Shockley [1] have been the first to approach the problem of the semiconductor with a nonuniform band structure. P. R. Emtage [2] has studied the photovoltaic effect in these semiconductors. The carrier distribution in graded band gap-semiconductors under asymmetric band edge gradients was analyzed by S. Chattopadhyaya and V. K. Mathur [3]. T. Gora and F. Williams [4] have approached the theory of the electronic states and transport in graded mixed semiconductors. A. Marshak and K. M. van Vliet have related in the semiconductors with nonhomogeneous band structure the density of states [5], the density of electrical current [6], the carriers density [7], the law of the junction [8] and the Shockley like equation [9]. Their fundamental result is

$$(1) \quad \mathbf{j}_n = eD_n \nabla n + en\mu_n \mathbf{E} - n\mu_n \nabla(\chi + \Gamma_n).$$

M. Stordeur and H. T. Langmuir [10] have analysed the anisotropy of transport properties in these semiconductors. In previous papers we have presented a theoretical study of the electron and phonon density of states [11], [12], the drag phenomenon [13] and the photoelectrolysis current [14] in semiconductors with position-dependent band structure.

In this paper we report a theoretical study of the Hall effect in degenerate and nondegenerate semiconductors of n -type, with nonuniform band structure, taking into account the drag of phonons by electrons and the phonon-phonon drag.

2. The Hall Electric Field. In a semiconductor with position dependent band structure, the Boltzmann equation for electrons, in the relaxation time approximation in the presence of the electric and magnetic field, is [6]

$$(2) \quad \frac{\partial f(\mathbf{k}, \mathbf{r}, t)}{\partial t} + \mathbf{v}_k \nabla f(\mathbf{k}, \mathbf{r}, t) - \frac{1}{\hbar} [\nabla E_c(\mathbf{r}) + \nabla W(\mathbf{r}) + e(\mathbf{v} \times \mathbf{B})] \nabla_{\mathbf{k}} f(\mathbf{k}, \mathbf{r}, t) = \frac{f - f_0}{\tau_e},$$

where $f_0(\mathbf{k}, \mathbf{r})$ is the electron equilibrium distribution, $f(\mathbf{k}, \mathbf{r}, t)$ is the non-equilibrium distribution, W is the electron kinetic energy and τ_e —the relaxation time for the scattering processes in the conduction band. In the non-equilibrium state, the electron distribution consists of an isotropic and an anisotropic part

$$(3) \quad f(\mathbf{k}, \mathbf{r}) = f_0(\mathbf{k}, \mathbf{r}) + f_{an}(\mathbf{k}, \mathbf{r}) = f_0(\mathbf{k}, \mathbf{r}) - k\chi(E, \mathbf{r}) \frac{\partial f_0(\mathbf{k}, \mathbf{r})}{\partial E}.$$

The electron scattering by lattice vibrations is almost elastic and the electrons interact only with longwave phonons. These phonons, known as electron-phonons, have a wave vector $p < 2k$, where k is the electron wave vector. The phonons for which the wave vector q satisfy the inequality $q > 2k$ are named thermic-phonons. In a semiconductor it is possible to exist a two-step drag: at the application of an electric field, the charge carriers drag the electron-phonons and this drag the thermic-phonons [15], [16]. In this case, the electron-phonons and thermic-phonons distribution are

$$(4) \quad N(\mathbf{p}, \mathbf{r}) = N_0(\mathbf{p}, \mathbf{r}) + \mathbf{p} \cdot \mathbf{A}(\mathbf{p}, \mathbf{r}) = N_0(\mathbf{p}, \mathbf{r}) - \mathbf{p} \cdot \mathbf{U}(p) \frac{\partial N_0}{\partial \omega_p},$$

$$(5) \quad F(\mathbf{q}, \mathbf{r}) = F_0(\mathbf{q}, \mathbf{r}) + F_{an}(\mathbf{q}, \mathbf{r}) = F_0(\mathbf{q}, \mathbf{r}) - \mathbf{q} \cdot \mathbf{U} \frac{\partial F_0}{\partial \omega_q}.$$

For a stationary state, the coupled Boltzmann equations for electrons, electron-phonons and thermic-phonons are:

$$(6) \quad \mathbf{v}_k \nabla f(\mathbf{k}, \mathbf{r}) - \frac{1}{\hbar} h(\nabla E_c + \nabla W) \nabla_k f(\mathbf{k}, \mathbf{r}) = I_{ep}\{f, N\} + I_{ea}\{f\},$$

$$(7) \quad c_s \nabla T \frac{\partial N(\mathbf{p}, \mathbf{r})}{\partial T} = I_{pe}\{N, f\} + I_{pT}\{N, F\} + I_{pa}\{N\},$$

$$(8) \quad c_s \nabla T \frac{\partial F(\mathbf{q}, \mathbf{r})}{\partial T} = I_{TP}\{F, N\} + I_{TT}\{F\} + I_{Ta}\{F\},$$

where I_{ep} , I_{ea} , I_{pe} s. o. are the scattering integrals. From (3) ... (8) result [15]

$$(9) \quad \mathbf{U} = L_{Tp}^{-1} L_T \langle \langle \mathbf{U}_{(p)} \rangle \rangle_n - s L_T T^{-1} \nabla T,$$

$$(10) \quad \mathbf{U}_{(p)} = L_{pe}^{-1} L_p \int_{\varepsilon(p/2)}^{\infty} \mathbf{V}(\varepsilon) \frac{\partial f_0}{\partial \varepsilon} d\varepsilon + L_{pT}^{-1} L_p \mathbf{U} - s L_p T^{-1} \nabla T.$$

In a nondegenerate semiconductor, the final states are practically empty. Thus

$$(11) \quad \begin{aligned} I_{ep}\{f, N\} = & B \sum_p \{ [f(\mathbf{k}, \mathbf{r} + \mathbf{p}, \mathbf{r})(N(\mathbf{p}) + 1) - f(\mathbf{k}, \mathbf{r})N(\mathbf{p})] \delta[E(\mathbf{k} + \mathbf{p}, \mathbf{r}) - \\ & - E(\mathbf{k}, \mathbf{r}) - \hbar\omega_p] + [f(\mathbf{k} - \mathbf{p}, \mathbf{r})N(\mathbf{p}) - f(\mathbf{k}, \mathbf{r})(N(\mathbf{p}) + 1)] \delta[E(\mathbf{k}, \mathbf{r}) - \\ & - E(\mathbf{k} - \mathbf{p}, \mathbf{r}) - \hbar\omega_p] \}, \end{aligned}$$

where

$$(12) \quad B = \frac{\pi C_1^2}{V \rho c_s};$$

C_1 is the total deformation potential, c_s — the longitudinal sound velocity; V — the volume of the crystal. Applying the method of N. Perrin and H. Budd [17], in the calculation of I_{ep} we consider only the linear terms in k and p . Thus the final expression is

$$(13) \quad I_{ep}\{f, N\} = k \left(\chi \frac{\partial f_0}{\partial E} + \frac{f_0 \mathbf{n}}{x} \right),$$

where $\mathbf{n}$ is the unit vector of $\mathbf{A}(\mathbf{p})$ and τ_a is the electron-acoustic phonon relaxation time when the anisotropic part of the distribution function is neglected;

$$(14) \quad \frac{1}{\tau_a} = \frac{C_1^2 m^*}{4\pi \rho c_s \hbar^3 T_e} \int_0^{2k} [2N_0(p) + 1] p^4 dp.$$

The second term of Eq. (13) is given by the anisotropic part of the phonon distribution

$$(15) \quad \frac{1}{x} = \frac{C_1^2 m^*}{4\pi \rho \hbar k_B T_e} \int_0^{2k} A(p) p^5 dp,$$

T_e being the electron temperature.

The term determined by the interaction of the electrons with defects is [18]

$$(16) \quad I_{ea}\{f\} = k \chi \frac{\partial f_0}{\partial E} \frac{1}{\tau_I},$$

where τ_I is the relaxation time of this interaction.

By using (13), (16) and (6), the Boltzmann equation becomes

$$(17) \quad \mathbf{v}_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} - \frac{1}{\hbar} e(\mathbf{v}_k \times \mathbf{B}) \nabla_k f = \\ = k \chi \frac{\partial f_0}{\partial E} \left(\frac{1}{\tau_a} + \frac{1}{\tau_I} \right) + \frac{k \mathbf{n} f_0}{x}.$$

From (3) we obtain

$$(18) \quad \nabla_k f = \nabla_k \left(f_0 - \frac{\partial f_0}{\partial E} \chi \mathbf{k} \right) = \frac{\partial f_0}{\partial E} \hbar \mathbf{v}_k - (\chi \mathbf{k}) \nabla_k \left(\frac{\partial f_0}{\partial E} \right) - \frac{\partial f_0}{\partial E} \nabla_k (\chi \cdot \mathbf{k}).$$

The first and second term of (18) multiplied by $(\mathbf{v} \times \mathbf{B})$ give zero, and thus it results

$$(19) \quad \frac{1}{\hbar} e(\mathbf{v}_k \times \mathbf{B}) \nabla_k f = - \frac{e(\mathbf{v}_k \times \mathbf{B})}{\hbar} \frac{\partial f_0}{\partial E} \nabla_k (\chi \cdot \mathbf{k}) = - \frac{e}{\hbar} (\mathbf{B} \times \chi) \frac{\partial f_0}{\partial E} \mathbf{v}_k.$$

Introducing (19) in (17) we have

$$(20) \quad \mathbf{v}_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} + \frac{e}{\hbar} \frac{\partial f_0}{\partial E} (\mathbf{B} \times \chi) \mathbf{v}_k = \mathbf{k} \chi \frac{\partial f_0}{\partial E} \frac{1}{\tau} + \frac{\mathbf{k} n f_0}{x},$$

where

$$(21) \quad \frac{1}{\tau} = \frac{1}{\tau_a} + \frac{1}{\tau_I}.$$

The vectorial Eq. (20) gives the expression of the factor χ , which determines the anisotropic part of the distribution function:

$$(22) \quad \begin{aligned} \chi = & \frac{1}{1 + \frac{\tau^2 e^2 B^2 v_k^2}{k^2 \hbar^2}} \left\{ \frac{\tau}{\mathbf{k}} \left[\mathbf{v}_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} - \frac{\mathbf{k} n f_0}{x} \right] + \right. \\ & + \frac{\tau^2 e}{\hbar} \frac{\mathbf{B} \mathbf{v}_k}{\mathbf{k}} \times \frac{1}{\mathbf{k}} \left[\mathbf{v}_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} - \frac{\mathbf{k} n f_0}{x} \right] + \\ & \left. + \frac{\tau^3 e^2}{k^2 \mathbf{k} \hbar} \frac{\partial f_0}{\partial E} \left[\left(\mathbf{v}_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} - \frac{\mathbf{k} n f_0}{x} \right) \mathbf{B} \mathbf{v}_k \right] \mathbf{B}_k \right\}. \end{aligned}$$

In a semiconductor with position-dependent band structure, the quasi-Fermi level satisfy the relation [5]

$$(23) \quad \nabla E_{fn} = e\mathbf{E} + \mathbf{C},$$

where $\mathbf{C}$ is the notation

$$(24) \quad \mathbf{C} = -\nabla \chi + \left(\frac{\partial E_{fn}}{\partial n} \right)_{T, \varepsilon} \nabla n + \left(\frac{\partial E_{fn}}{\partial T} \right)_{n, \varepsilon} \nabla T - \left(\frac{\partial E_{fn}}{\partial n} \right)_{T, \varepsilon} \frac{1}{4\pi^3} \int d^3k \frac{\partial f_0}{\partial E} \nabla W.$$

Considering that in the Hall effect no temperature gradient appears and using (23) the relation (22) may be written as

$$(25) \quad \chi = b_1 \mathbf{E} + b_2 \mathbf{B} \times \mathbf{E} + b_3 (\mathbf{E} \times \mathbf{B}) \mathbf{B} + \mathbf{D},$$

where

$$(26) \quad b_1 = \frac{\tau e v_k}{\left(1 + \frac{\tau^2 e^2 B^2 v_k^2}{k^2 \hbar^2} \right) \mathbf{k} \frac{\partial f_0}{\partial E}},$$

$$(27) \quad b_2 = \frac{\tau^2 e^2 \mathbf{v}_k}{\left(1 + \frac{\tau^2 e^2 B^2 v_k^2}{k^2 \hbar^2} \right) \mathbf{k} \hbar \frac{\partial f_0}{\partial E}},$$

$$(28) \quad b_3 = \frac{\tau^3 e^3 p_k^2 v_k}{\left(1 + \frac{\tau^2 e^2 B^2 v_k^2}{k^2 h^2}\right) k^2 h^2 \frac{\partial f_0}{\partial E} k}$$

$$(29) \quad \mathbf{D} = \frac{1}{1 + \frac{\tau^2 e^2 B^2 v_k^2}{k^2 h^2}} \left\{ \frac{\tau}{k} \frac{\partial f_0}{\partial E} \left(\mathbf{v}_k \mathbf{C} \frac{\partial f_0}{\partial E} - \frac{\mathbf{k} n f_0}{x} \right) + \frac{\mathbf{v}_k \tau^2 e \mathbf{v}_k}{k h} \frac{\partial f_0}{\partial E} \mathbf{B} \right. \\ \left. \left(\mathbf{C} \frac{\partial f_0}{\partial E} - \frac{\mathbf{k} n f_0}{\mathbf{v}_k x} \right) + \frac{\tau^3 e^2}{k^2 k h^2} \frac{\partial f_0}{\partial E} \left[\left(\mathbf{v}_k \mathbf{C} \frac{\partial f_0}{\partial E} - \frac{\mathbf{k} n f_0}{x} \right) \mathbf{B} \mathbf{v}_k \right] \mathbf{B} \mathbf{v}_k \right\}.$$

In the Hall effect $\mathbf{E} \perp \mathbf{B}$. We consider that $\mathbf{B}$ is oriented along Oz axis: $\mathbf{B}(0,0,B)$, $\mathbf{E}(E_x, E_y, 0)$. Relation (22) thus becomes

$$(30) \quad \chi = i(b_1 E_x - b_2 B E_y + D_x) + j(b_1 E_y + b_2 B E_x + D_y) + k D_z.$$

The current density is [18]

$$(31) \quad \mathbf{j} = -\frac{ne}{h} \langle \chi \rangle.$$

Substituting (30) in (31) there results

$$(32) \quad j_x = a_1 E_x - a_2 E_y + d_1,$$

$$(33) \quad j_y = a_2 E_x + a_1 E_y + d_2;$$

a_1, a_2, d_1 and d_2 are symbols for

$$(34) \quad a_1 = -\frac{ne}{h} \langle b \rangle, \quad a_2 = -\frac{ne}{h} \langle b_2 B \rangle,$$

$$(35) \quad d_1 = -\frac{ne}{h} \langle D_x \rangle, \quad d_2 = -\frac{ne}{h} \langle D_y \rangle.$$

From (32) and (33) with the characteristic condition of the Hall effect, $j_x = j_y = 0$, we obtain the following electric field intensity:

$$(36) \quad E_y = -\frac{a_2}{a_1^2 + a_2^2} \left(j + \frac{a_1 d_2}{a_2} + d_1 \right).$$

For a degenerate semiconductor we have [18]

$$(37) \quad I_{ep} \{f, N\} = \sum_p \{ W^-(\mathbf{k} + \mathbf{p}, \mathbf{k}) f(\mathbf{k} + \mathbf{p}, \mathbf{r}) [1 - f(\mathbf{k}, \mathbf{r})] + \\ + W^+(\mathbf{k} - \mathbf{p}, \mathbf{k}) f(\mathbf{k} - \mathbf{p}, \mathbf{r}) [1 - f(\mathbf{k}, \mathbf{r})] - W^+(\mathbf{k}, \mathbf{k} + \mathbf{p}) f(\mathbf{k}, \mathbf{r}) \cdot \\ \cdot [1 - f(\mathbf{k} + \mathbf{p}, \mathbf{r})] - W^-(\mathbf{k}, \mathbf{k} - \mathbf{p}) f(\mathbf{k}, \mathbf{r}) [1 - f(\mathbf{k} - \mathbf{p}, \mathbf{r})] \}.$$

Calculating this collision integral results

$$(38) \quad I_{sp}\{f, N\} = k \left[\chi \frac{\partial f_0}{\partial E} - \frac{f_0(1-f_0)n}{x} \right],$$

where

$$(39) \quad \frac{1}{\tau_d} = \frac{1}{\tau_a} + \frac{1}{\tau_s},$$

$$(40) \quad \frac{1}{\tau_s} = \frac{8C_1^2 f_0(k, r) [1 - f_0(k, r)] m^* k^2}{V \rho k_s T \hbar}.$$

Following the same method as for a nondegenerate semiconductor, we have obtained χ and then E_y :

$$(41) \quad \begin{aligned} \chi = & \frac{1}{1 + \frac{\tau^2 e^2 B^2 p^2}{k^2 \hbar^2}} \left\{ \frac{\tau}{k} \frac{\partial f_0}{\partial E} \left[v_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} - \frac{k n f_0 (1 - f_0)}{x} \right] + \right. \\ & + \frac{\tau^2 e}{\hbar} \frac{B v_k}{k} \times \frac{1}{k} \left[v_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} - \frac{k n f_0 (1 - f_0)}{x} \right] + \\ & \left. + \frac{\tau^3 e^2}{k^2 k \hbar} \frac{\partial f_0}{\partial E} \left[\left(v_k \left(\nabla E_{fn} + \frac{E - E_{fn}}{T} \nabla T \right) \frac{\partial f_0}{\partial E} - \frac{k n f_0 (1 - f_0)}{x} \right) B v_k \right] B v_k \right\}, \end{aligned}$$

where

$$(42) \quad \frac{1}{\tau} = \frac{1}{\tau_d} + \frac{1}{\tau_I}.$$

E_y is given by the same general relation (36), but in the formulas for d_1 and d_2 the term with x contains also the factor $1 - f_0$.

3. Conclusions. The relation (36) is the general expression of the Hall electric field intensity in an atomic semiconductor of n -type with position-dependent band structure, taking into account the phonon-phonon drag and the electron drag. If the semiconductor is homogeneous and the drag phenomena are not taken into account, d_1 , d_2 are zero and from (36) we obtain the known relation for E_y .

Received May 9, 1990

Polytechnic Institute, Iassy,
Department of Physics

REFERENCES

1. Bardeen J., Shokley W., Phys. Rev., **80**, 72 (1950).
2. Emtage P. R., J. Appl. Phys., **33**, 1962 (1950).
3. Chattopadhyaya S., Mathur V., Phys. Rev., **B9**, 8, 3 517 (1974).
4. Gora T., Williams F., Phys. Rev., **177**, 3 (1969).
5. Marshak A. H., van Vliet K. M., Phys. St. Sol. (b), **117**, 329, 1 (1980).

6. Marshak A. H., van Vliet K. M., Solid St. Electron., **21**, 417 (1979).
7. Marshak A. H., van Vliet K. M., Solid St. Electron., **22**, 567 (1979).
8. Marshak A. H., van Vliet K. M., Solid St. Electron., **21**, 4 291 (1978).
9. Marshak A. H., van Vliet K. M., Solid St. Electron., **23**, 49 (1980).
10. Stordeur M., Langhammer H., Phys. St. Sol. (b), **101**, 2, 525 (1980).
11. Zet Gh., Istrate M., Bul. Inst. Polit. Iași, **XXIX(XXXIII)**, 1-4, s. I, 9 (1983).
12. Istrate M., Carasevici C., Istrate S., Bul. Inst. Polit. Iași, **XXXIV(XXXVIII)**, 1-4, s. I, 85 (1988).
13. Istrate M., Rev. Roum. de Phys., **28**, 2, 159 (1983).
14. Istrate M., Zet Gh., Istrate S., Neagu E., Intern. Conf. Photovoltaic and Optoelectr. Processes, Bucur. și I. 1984, p. 13.
15. Belcik A., Kozlov A. B., F.T.P., **20**, 1, 53 (1986).
16. Bochkov A. V., Gurevici O., Mashevici O., F.T.P., **20**, 3, 572 (1986).
17. Perrin N., Budd H., Phys. Rev., **B9**, 8, 3 454 (1972).
18. Anselm A. H., *Introduction to Semiconductor Theory*. Mir Publishers, Moscow, 1982.

FENOMENUL DE ANTRENARE ȘI EFECTUL HALL ÎN SEMICONDUCTORI CU BENZI NEUNIFORME

(Rezumat)

Se studiază intensitatea cîmpului electric Hall într-un semiconductor atomic, de tip- n , cu structură de bandă dependentă de poziție. Se ia în considerare antrenarea fononilor de către electroni și antrenarea fonon-fonon, analizîndu-se atît cazul semiconductoarelor degenerați cît și al celor nedegerați.

The first of these is the fact that the
the second is the fact that the
the third is the fact that the
the fourth is the fact that the
the fifth is the fact that the
the sixth is the fact that the
the seventh is the fact that the
the eighth is the fact that the
the ninth is the fact that the
the tenth is the fact that the
the eleventh is the fact that the
the twelfth is the fact that the
the thirteenth is the fact that the
the fourteenth is the fact that the
the fifteenth is the fact that the
the sixteenth is the fact that the
the seventeenth is the fact that the
the eighteenth is the fact that the
the nineteenth is the fact that the
the twentieth is the fact that the
the twenty-first is the fact that the
the twenty-second is the fact that the
the twenty-third is the fact that the
the twenty-fourth is the fact that the
the twenty-fifth is the fact that the
the twenty-sixth is the fact that the
the twenty-seventh is the fact that the
the twenty-eighth is the fact that the
the twenty-ninth is the fact that the
the thirtieth is the fact that the
the thirty-first is the fact that the
the thirty-second is the fact that the
the thirty-third is the fact that the
the thirty-fourth is the fact that the
the thirty-fifth is the fact that the
the thirty-sixth is the fact that the
the thirty-seventh is the fact that the
the thirty-eighth is the fact that the
the thirty-ninth is the fact that the
the fortieth is the fact that the
the forty-first is the fact that the
the forty-second is the fact that the
the forty-third is the fact that the
the forty-fourth is the fact that the
the forty-fifth is the fact that the
the forty-sixth is the fact that the
the forty-seventh is the fact that the
the forty-eighth is the fact that the
the forty-ninth is the fact that the
the fiftieth is the fact that the
the fifty-first is the fact that the
the fifty-second is the fact that the
the fifty-third is the fact that the
the fifty-fourth is the fact that the
the fifty-fifth is the fact that the
the fifty-sixth is the fact that the
the fifty-seventh is the fact that the
the fifty-eighth is the fact that the
the fifty-ninth is the fact that the
the sixtieth is the fact that the
the sixty-first is the fact that the
the sixty-second is the fact that the
the sixty-third is the fact that the
the sixty-fourth is the fact that the
the sixty-fifth is the fact that the
the sixty-sixth is the fact that the
the sixty-seventh is the fact that the
the sixty-eighth is the fact that the
the sixty-ninth is the fact that the
the seventieth is the fact that the
the seventy-first is the fact that the
the seventy-second is the fact that the
the seventy-third is the fact that the
the seventy-fourth is the fact that the
the seventy-fifth is the fact that the
the seventy-sixth is the fact that the
the seventy-seventh is the fact that the
the seventy-eighth is the fact that the
the seventy-ninth is the fact that the
the eightieth is the fact that the
the eighty-first is the fact that the
the eighty-second is the fact that the
the eighty-third is the fact that the
the eighty-fourth is the fact that the
the eighty-fifth is the fact that the
the eighty-sixth is the fact that the
the eighty-seventh is the fact that the
the eighty-eighth is the fact that the
the eighty-ninth is the fact that the
the ninetieth is the fact that the
the ninety-first is the fact that the
the ninety-second is the fact that the
the ninety-third is the fact that the
the ninety-fourth is the fact that the
the ninety-fifth is the fact that the
the ninety-sixth is the fact that the
the ninety-seventh is the fact that the
the ninety-eighth is the fact that the
the ninety-ninth is the fact that the
the hundredth is the fact that the

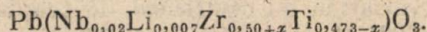
**LES EFFET DES VARIATIONS DE LA TEMPÉRATURE SUR LE
COMPORTEMENT PIÉZOÉLECTRIQUE DES ZIRCO-TITANATES
DE PLOMB CÉRAMIQUES AVEC DES MODIFICATEURS,
DONT LE RAPPORT Zr/Ti VARIE DANS LE DOMAINE
DE LA TRANSITION MORPHOTROPIQUE DE PHASE**

PAR

IRINA ANTONESCU, M. ANTONESCU, C. TĂNĂȘOIU et POMPILIU NICOLAU

1. Introduction. On a réalisé quatre matériaux céramiques nouveaux à forte activité piézoélectrique, dont le comportement à la température ambiante a été présenté dans [1]. Les valeurs élevées de leurs températures Curie ($>380^{\circ}\text{C}$) justifient l'intérêt pour leur comportement à des températures supérieures à 80°C , donc l'investigation d'un intervalle élargi de températures ($-25^{\circ}\text{C} \dots +400^{\circ}\text{C}$) par rapport à celui usuel pour les applications, $-20^{\circ}\text{C} \dots +80^{\circ}\text{C}$.

2. Matériaux. Les quatre matériaux nouveaux ont été conçus de sorte qu'on puisse mettre en évidence l'influence de la variation du quotient Zr/Ti dans le zirco-titanate de plomb modifié par la substitution partielle des ions de Ti^{4+} par des ions de Nb^{5+} , la compensation des valences étant réalisée à l'aide des ions de Li^{+}



Les composés ayant les valeurs du quotient Zr/Ti 1,057 ; 1,102 ; 1,148 et, respectivement, 1,196 ont été nommés ZrTiO, ZrTi1, ZrTi2 et, respectivement, ZrTi3.

On a constaté, par diffraction des rayons X, que les matériaux sont monophasiques et que leur tétragonalité diminue rapidement à mesure que la valeur du quotient Zr/Ti augmente. Les observations métallographiques ont évincé l'amoindrissement des cristallites à mesure que la valeur du quotient Zr/Ti s'accroît ; de sorte, le composé ZrTiO a des cristallites dont les dimensions dépassent $7\mu\text{m}$, tandins que celles de ZrTi3 sont inférieures à $3\mu\text{m}$.

On a mesuré les principales grandeurs diélectriques et piézoélectriques dans l'intervalle $-25^{\circ}\text{C} \dots +400^{\circ}\text{C}$ à l'aide d'un thermostat dont la stabilité et la précision sont meilleures que 1°C dans tout l'intervalle [2].

3. Résultats et discussions. Dans la Fig. 1 on a représenté les variations du coefficient de couplage planaire par rapport aux variations de la température. On y voit qu'au cas des matériaux qui contiennent 50% et, respectivement, 51% ions de Zr sur les sites octaédriques, le coefficient k_p diminue de façon monotone quand la température augmente, sa variation

relative dans l'intervalle $-20^{\circ}\text{C} \dots +80^{\circ}\text{C}$ ne dépassant pas 2,5%. Ce comportement prouve l'absence des tensions mécaniques internes résiduelles dans le domaine des températures investiguées, le degré d'alignement des domaines ferroélectriques étant maximal à partir des températures inférieures à 25°C .

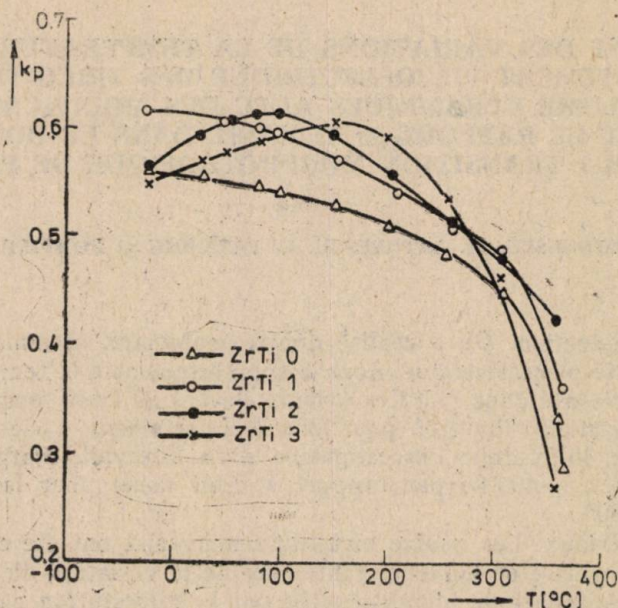


Fig. 1

Par centre, la variation du coefficient de couplage planaire k_p au cas des matériaux ayant 52% et, respectivement, 53% ions de Zr n'est pas monotone, les maximums étant situés à 100°C et, respectivement, à 150°C . Cette dernière valeur pourrait être expliquée tant par la granulation très fine du matériau ZrTi3 , dont les cristallites ont des dimensions inférieures à $3 \mu\text{m}$, que par la tétragonalité réduite de ce matériau, qui a besoin d'énergies plus grandes pour l'accroissement du degré d'alignement des domaines ferroélectriques. Les variations relatives du coefficient de couplage planaire sont assez grandes, dans l'intervalle $-20^{\circ}\text{C} \dots +80^{\circ}\text{C}$: 8,4% et, respectivement, 5,3%.

En ce que concerne la constante de fréquence dans le mode radial de vibration N_p au cas des matériaux ZrTi0 et ZrTi1 elle présente un comportement monotone similaire à celui du coefficient de couplage planaire, tandis qu'au cas des matériaux ZrTi2 et ZrTi3 elle présente des minimums aux températures de 100°C et, respectivement, de 200°C (Fig. 2).

Les coefficients de température de la fréquence de résonance au cas de ZrTi0 et ZrTi1 , calculés pour le domaine $-20^{\circ}\text{C} \dots +80^{\circ}\text{C}$, sont assez réduits en pensant qu'il s'agit de matériaux à forte activité piézoélectrique, 90 ppm/ $^{\circ}\text{C}$ et 100 ppm/ $^{\circ}\text{C}$. Mais quand le pourcentage de Zr dépasse 51%, l'instabilité de la fréquence de résonance dans le mode radial de vibration va s'accroître : 840 ppm/ $^{\circ}\text{C}$ et, respectivement, 530 ppm/ $^{\circ}\text{C}$.

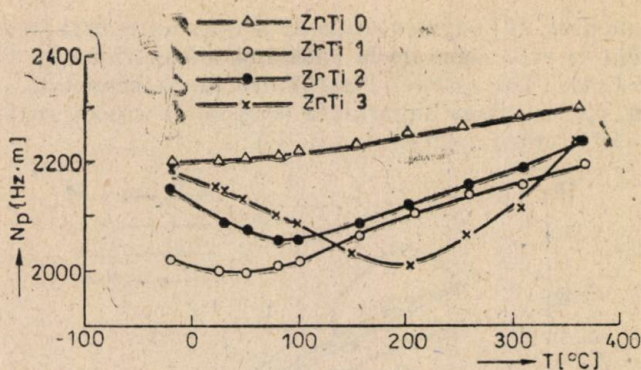


Fig. 2

La Fig. 3 représente les variations de la constante piézoélectrique de charge d_{31} , par rapport aux variations de la température. Ce qu'on peut remarquer c'est la constance des valeurs de cette grandeur jusqu'à environ 200°C, au cas des matériaux ZrTi0 et ZrTi1. C'est le résultat des variations monotones, dans des sens contraires, du coefficient de couplage planaire et, respectivement, de la constante de fréquence, toutes les deux corrélées avec une faible variation de la constante diélectrique dans cet intervalle de température, assez éloigné de la température Curie de ces matériaux (396°C). En ce que concerne les matériaux ZrTi2 et ZrTi3, leurs constantes de charge d_{31} présentent des maximums assez plats, à partir de températures inférieures à la température Curie (plus basses de 75°C et, respectivement, de 130°C que celle-ci). Ce fait prouve que l'amointrissement du coefficient de couplage

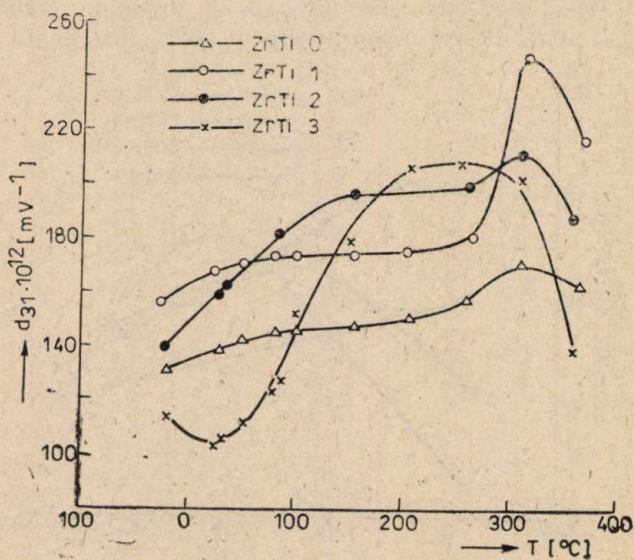


Fig. 3

planaire, d'une part, et l'augmentation de la constante de la fréquence, d'autre part, dominant la croissance de la constante diélectrique au voisinage de la température Curie. Par contre, l'instabilité de la constante piézoélectrique de la tension, g_{31} , est assez importante (Fig. 4) au cas de tous les matériaux ($-12\% \dots -17\%$ entre -20°C et $+80^\circ\text{C}$).

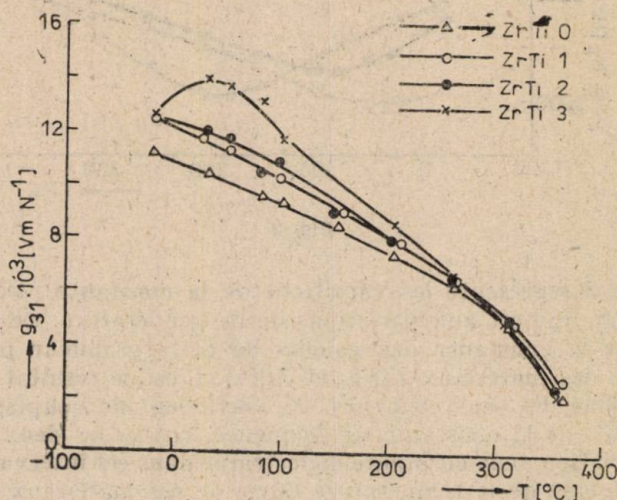


Fig. 4

La compliance élastique s_{11}^E varie différemment par rapport à la température, quand le pourcentage de Zr est différent (Fig. 5) : pour 50% et 51% ions de Zr la compliance élastique diminue faiblement à mesure que la température s'accroît (les variations relatives dans le domaine $-20^\circ\text{C} \dots +80^\circ\text{C}$ ne dépassent 3%) ; pour 52% et, respectivement, 53% ions de Zr, la compliance

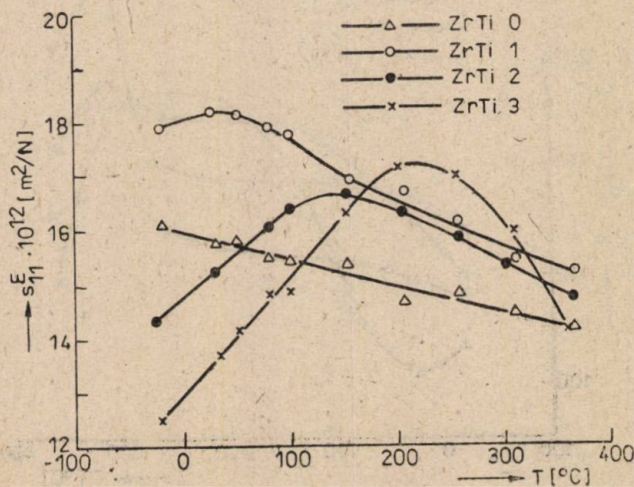


Fig. 5

élastique présente des maximums assez bien exprimés, ses variations relatives dans le domaine $-20^{\circ}\text{C} \dots +80^{\circ}\text{C}$ étant assez importantes (11% et 16%). Ces maximums correspondent aux minimums de la constante de la fréquence dans le mode radial de vibration.

4. Conclusions. 1. On a étudié le comportement piézoélectrique de quatre matériaux céramiques à base de zirco-titanate de plomb modifié par Nb, dont le quotient Zr/Ti est variable dans les limites de la transition morphotropique de phase, à des températures comprises entre -25°C et 400°C .

2. Les matériaux avec 50% et, respectivement, 51% ions de Zr présentent un comportement différent de celui présenté par les matériaux avec 52% et, respectivement, 53% ions de Zr :

a) les premiers ont des coefficients de couplage planaire et des constantes de la fréquence dans le mode radial de vibration qui varient de façon monotone quand la température s'accroît. Ce fait prouve l'absence des tensions mécaniques internes résiduelles à partir de températures basses ;

b) les derniers ont un comportement caractérisé par la présence des maximums du coefficient de couplage planaire et, respectivement, des minimums de la constante de la fréquence dans le mode radial de vibration. Ce sont les effets d'une tétragonalité réduite et d'une granulation très fine, spécialement au cas du pourcentage de 53% ions de Zr sur les sites octaédriques.

3. La variation de la constante piézoélectrique de charge d_{31} est le résultat de la compétition entre les variations du coefficient de couplage planaire, de la constante de la fréquence dans le mode radial de vibration et de la constante diélectrique libre. Les matériaux à faible pourcentage de Zr ont des constantes d_{31} plus stables que les autres.

4. La stabilité de la constante piézoélectrique de la tension g_{31} est réduite au cas de tous les matériaux du système.

5. Les variations de la compliance élastique s_{11}^E peuvent être corrélées avec celles de la constante de la fréquence dans le mode radial de vibration. De même, les composés avec moins de Zr ont des compliances élastiques plus stables.

6. Fondé sur tous ces résultats, on peut recommander les matériaux ZrTiO et ZrTi1 pour l'utilisation dans un domaine de températures plus large que $-20^{\circ}\text{C} \dots +80^{\circ}\text{C}$, leurs paramètres présentant une stabilité améliorée par rapport aux variations de la température que ceux d'autres matériaux céramiques à forte activité piézoélectrique [3].

Reçu le 22 Novembre, 1990

Institut Polytechnique, Jassy
et
Institut de Physique et de Technologie
des Matériaux, Bucharest

BIBLIOGRAPHIE

1. Jemna I., Luca E., Nicolau P., Tănăsioiu C., Niculescu H., Les travaux du Symposium National de Technologie Électronique et Fiabilité, Jassy, 1983, t. I, pp. 103-108.
2. Jemna I., Thèse de doctorat, Centre de Physique Technique, Jassy, 1984.
3. Antonescu I., Antonescu M., Tănăsioiu C., Nicolau P., Bul. Inst. Polit. Iași, XXXV(XXXIX), 1-2, s. I, 95-99 (1989).

EFECTELE VARIAȚIILOR TEMPERATURII ASUPRA COMPORTĂRII
PIEZOELECTRICE A ZIRCO-TITANAȚILOR DE PLUMB MODIFICAȚI,
AL CĂROR RAPORT Zr/Ti VARIAZĂ ÎN LIMITELE TRANZIȚIEI
MORFOTROPICE DE FAZĂ

(Rezumat)

S-a studiat comportarea piezoelectrică, la temperaturi cuprinse între -20°C și $+400^{\circ}\text{C}$, a patru materiale ceramice noi, de tip zirco-titanat de plumb modificat cu Nb, în care raportul Zr/Ti variază în limitele tranziției morfotropice de fază. S-a corelat modul de variație a coeficientului de cuplaj planar, a constantei frecvenței în modul de vibrație radial și a constantei piezoelectrice de sarcină cu procentul ionilor de Zr de pe locurile octaedrice. Dintre materialele studiate s-au evidențiat compuşii $ZrTiO$ și $ZrTi1$, cu o stabilitate deosebită a parametrilor piezoelectrici chiar la temperaturi superioare limitei domeniului uzual pentru aplicații, -20°C ... $+80^{\circ}\text{C}$.

40822